Computation of the Alexander Polynomial of a Plane Curve Singularity and an Implementation in SINGULAR

Fernando Hernando

Dissertation submitted for the Degree Master of Science at the
Department of Mathematics, University of Kaiserslautern
Kaiserslautern, September 2002

Supervised by Gert-Martin Greuel and Félix Delgado

To my parents,
to my brother and sisters,
to my uncle and aunt,
and to Eva.

Acknowledment

I want to express my deep gratitude to Dr. Thomas Keilen, my supervisor for this thesis, who devoted a great part of his value time and gave very important suggestions and orientation to this work to be successfully done.

I am specially grateful to Prof Dr. G.M. Greuel and Prof Dr Antonio Campillo for their continued guidance throughout this program and their concern in my studies, and for giving me a topic that is not only interesting but also challenging and manageable within the time frame. Special thanks goes to Dr. Felix Delgado for his value suggestions in no small way to improve this work. Furthermore I am extremely grateful to Dr. Ana Nuñez for her confidence and help.

I would like to take this opportunity to deliver my sincere thank to all the members of the Department of Mathematics, Universidad de Valladolid and Universit$\ddot{a}$t Kaiserslautern for their tremendous help given to me in my career.

Finally, I would like to thank the University of Bayreuth and the European research network in Algebraic Geometry EAGER, which made my studies possible with their financial support.

Contents

0.1 Introduction

With the aid of this work we try to clarify the classification of plane curves singularities with respect to topological equivalence. In order to do this we will introduce some invariants which will allow us to distinguish the singularity types. There exist many classical invariants, for example: the Puiseux pairs, the multiplicity sequence, the characteristic exponents, the resolution graph or the Alexander polynomial, all of which are equivalent and are used by important authors. It is possible to find them in the papers from Campillo-Delgado-Gusein, Eisenbud-Neumann, Theo de Jong-Pfister, Greuel-Lossen-Shustin, Brieskorn-Knörrer, Zariski.

Our purpose is to define, explain and compute the resolution graph and the Alexander polynomial. For this we first start in Chapter 1 fixing notions, i.e. what we mean by a plane curve singularity, a topological invariant, and what we actually mean by topologically equivalence. Moreover, in this chapter we describe an algorithm to get a Puiseux parametrization, this algorithm consists basically in the calculation of the y-roots of a given $f \in \mathbb{C}\{x, y\}$. This parametrization will be very useful to get the characteristic exponents.

In Chapter 2 we are going to study irreducible plane curves singularities. We start defining some invariants. Here is the first time we define the generators of the semigroup, the characteristic exponents and the Puiseux pairs, and closed formulas to get one of them from the others. As we have already said with a Puiseux parametrization it is easy to calculate the characteristic exponents and now with the formula which we give in this chapter we can deduce quickly the generators of the semigroup. Those play an important role in the irreducible case because the semigroup is finitely generated and it is possible to compute the Alexander polynomial from the generators of the semigroup.

In the same chapter we introduce the standard resolution of an irreducible plane curve singularity. This is a process consisting in blowing-up the curve singularity several times until the singularity becomes smooth, intersects the exceptional divisor transversal and only intersects it in one component. Associated to the standard resolution we define the multiplicity sequence and the resolution graph, that is the dual graph of the standard resolution. Finally we see that all the invariants determine each other and the topological type.

Once we have described the irreducible case, the reducible case is a generalization and is considered in Chapter 3. The most important part in this chapter is the theorem where we see that the resolution graph of a reducible reduced plane curve singularity can be calculated from the resolution graphs of its branches together with the contact numbers of the branches. The proof of this theorem is an algorithm that will be used to perform the incidence matrix in our procedure.

In Chapter 4 we define the Alexander polynomial of the reducible reduced plane curve singularity in terms of the extended semigroup, but this definition it is not useful for computation. So we show an equivalent formula given by Eisenbud-Neumann which expresses the Alexander polynomial in terms of the resolution graph. At the end we show the most classical definition of the Alexander polynomial and the fact that the zeta-function may be derived from the Alexander polynomial just replacing all the variables by t.

Until here we have collected the relevant theory, from the authors which we have already cited. Now in Chapter 5 we treat the implementation of procedures to compute the resolution graph and the Alexander polynomial. Those procedures are done following the algorithm described in Theo de Jong-Pfister book and in Campillo-Delgado-Zade papers respectively, and we have needed the aid of the library hnoether.lib by Martin Lamm to calculate some invariants that he had already calculated in SINGULAR. Our work have been the implementation in SINGULAR. While we were doing the procedures we needed two extra procedures and we have included them in the thesis because they may be useful for the user. One of them calculates the characteristic exponents of an irreducible plane curve singularity from the multiplicity sequence, and the other calculates an irreducible polynomial defining an irreducible plane curve singularity with given characteristic exponents.

To show how to use the library alexpoly.lib in SINGULAR we have done an Appendix. There we explain which inputs are allowed, which commands are needed to execute the procedures and we explain the outputs.

In what follows we will suppose $\mathbb{C}$ to be the complex field and as customary $\mathbb{C}\{x,y\}$ the ring of convergent power series. And by $\mathbb{C}\langle\langle x\rangle\rangle$ we denote the field of fractionary power series.

Chapter 1

Introduction to the first notions

All the statements and proofs proposed in this chapter can be found in the book of Casas Alvero E. (see reference [6]).

1.1 Plane curves singularities

In the present section we want to introduce and study first properties of plane curves singularities, that is properties of plane curves locally at a singular point. Choosing suitable coordinates we may always suppose that the point in question is the origin in $\mathbb{C}^2$ and that the curve on some sufficiently small neighbourhood of zero is given as the zero set of a convergent power series.

Let us now give the notion of ”sufficiently small” a better meaning.

Definition 1.1.1. *Let X be a topological space and let $0 \in X$ be a point. Subsets A and B of X are called equivalent at 0 if there exists an open neighborhood U of 0 such that $A \cap U = B \cap U$. It is easy to check that this is indeed an equivalence relation. The equivalence class of A at 0 we call* ***the germ of A at*** 0*. A is called a representative of the germ. We write (A,0) for the germ of A at 0.*

Now we are ready to define what a germ of a plane curve is.

Definition 1.1.2. *Let $f \in \mathbb{C}\{x, y\} \setminus \mathbb{C}$. We call $\mathbb{C}\{x, y\}/(f)$ the* ***local ring*** *associated to f, and we denote by $\mathfrak{m}$ its* ***maximal ideal*** *or also will denote it by $\mathfrak{m}_C$, when we will introduce the notion of plane curve singularity (C,0).*

Consider an open neighborhood U of 0 on which f converges and the subset $V(f) := \{q \in U \mid f(q) = 0\}$ of U. We call, to the germ of a topological space

*$(V(f),0)$ together with the ring $\mathbb{C}\{x,y\}/(f)$, the **germ of the plane curve**, or **the plane curve singularity**, defined by f.*

We also write $(C_f,0)$ instead of $(V(f),0)$, respectively (C,0) if no confusion may arise, and by abuse of notation we call $f \in \mathbb{C}\{x,y\}$ a local equation for the plane curve singularity $(C,0) \subset (\mathbb{C}^2,0)$. Moreover, we usually denote by $\mathcal{O}_C$=$\mathbb{C}\{x,y\}/(f)$ the local ring of f.

If $f = f_1^{n1}\cdots f_r^{nr}$, is the irreducible decomposition of $f \in \mathbb{C}\{x,y\}$ then

$$V(f) = V(f_1^{n1}) \cup ... \cup V(f_r^{nr})$$

and we call $(V(f_i),0)$ a branch of $(V(f),0)$ and $n_i > 0$ the multiplicity of the branch, that is how many times it is the factor f_i in the equation of f.

Until now we have talked about plane curve singularity at the point 0. We now want to define when the plane curve singularity is called singular, for this we need first to introduce another important concept.

Definition 1.1.3. *Assume that $(C,0)$ is a plane curve singularity and $f \in \mathbb{C}\{x,y\}$ it is equation. We will denote by **mult(C,0)** the **multiplicity** of the plane curve singularity that may be defined by the relation $f \in \mathfrak{m}^{mult(C,0)} \setminus \mathfrak{m}^{mult(C,0)+1}$, i.e.*

$$mult(C,0) = min\{i+j \mid a_{i,j} \neq 0\},$$

where $f = \sum_{i+j\geq 0} a_{ij}x^iy^j$.

This allow us give the following definition.

Definition 1.1.4. *A plane curve singularity (C,0), is called **singular** if and only if $mult(C,0) > 1$. Then its representatives are said to have a singular point or a multiple point at 0.*

In what follows we shall derive a criterion for singularity. The most important criterion characterizes singular points as those at which the partial derivatives in the point vanish, we shall make this precise.

Proposition 1.1.5. *Let $(C,0)$ be a plane curve singularity defined by the equation $f = 0$ then*

$$mult(C,0) > 1 \Leftrightarrow \frac{\partial f(x,y)}{\partial x}(0,0) = \frac{\partial f(x,y)}{\partial y}(0,0) = f(0,0) = 0$$

Proof: Consider $f = \sum_{i,j} a_{i,j} x^i y^j$. If we see it as a power series expansion we have that $a_{1,0} = \frac{\partial f}{\partial x}(0,0)$ and $a_{0,1} = \frac{\partial f}{\partial y}(0,0)$. Thus we get $mult(C) > 1$ if and only if $a_{1,0} = a_{0,1} = 0$, and that finishes the proof. □

We are interested in the topological classification of the plane curve singularities, that is we want to capture the local qualitative properties of plane curves up to topological equivalence. Let us define what this means.

Definition 1.1.6. *Let $(C_1, 0)$ and $(C_2, 0)$ be two plane curves singularities. We say that they are **topologically equivalent**, and we denote it by $(C_1, 0) \sim_t (C_2, 0)$, if there exist, open neighborhoods U and V of zero, a homeomorphism $\Psi : U \to V$, and representatives C_i of $(C_i, 0)$,i=1,2, such that:*

$$\Psi(C_1 \cap U) = C_2 \cap V.$$

The tools to make such a classification are the invariants.

Definition 1.1.7. *A **topological invariant** is a map from the set of plane curves singularities to some set M (e.g. $\mathbb{Z}^k$)*

$$i : \{(C_f, 0) \mid f \in \mathbb{C}\{x, y\} \setminus \mathbb{C}\} \to M$$

such that $i(C_f, 0) = i(C_g, 0)$ whenever $(C_f, 0) \sim_t (C_g, 0)$.

Remark 1.1. We are going to study topological invariants of plane curves singularities. If two plane curves singularities are topologically equivalent then they have the same invariants. It would be very nice if the opposite would hold as well, that is, if we could find invariants, say i, such that $i(C_f, 0) = i(C_g, 0)$ implies $(C_f, 0) \sim_t (C_g, 0)$.

In this section we have developed a few general concepts with which we can investigate the zero set of analytic functions. We now want to describe the zero set of f, i.e. the set of solutions of the equation $f(x, y) = 0$, in a suitable neighborhood of the origin. We shall see through the next section that in a certain sense one can solve this equation quite explicitly.

1.2 Puiseux parametrization

In this section we are introducing parametrizations of plane curves singularities. In particular we describe the Newton-Puiseux algorithm for calculating the Puiseux expansion, a particular kind of parametrizations. This will be used in order to introduce the Characteristic exponents of the plane curve singularity, a particular invariant which turns out to characterize the topological equivalence class of the plane curve singularity .

We first start with a general definition of parametrization.

Definition 1.2.1. *Let* $(C, 0) \subset (\mathbb{C}^2, 0)$ *be a plane curve singularity, with equation* $f \in \mathbb{C}\{x, y\}$*. A* ***parametrization*** *of (C,0) at the point* 0 *(with respect to the coordinates x,y) is a pair of series* $y_1, y_2 \in \mathbb{C}\{t\}$ *such that:*

(1) $y_1(0) = 0, y_2(0) = 0.$

(2) y_1, y_2 *are not both in* $\mathbb{C}$*.*

(3) $f(y_1, y_2) = 0$ *as an element of* $\mathbb{C}\{t\}$*.*

Example 1.1. Let (C,0)=(V(f),0) $\subset$ ($\mathbb{C}^2$,0) be a plane curve singularity defined by $f = y^2 - x^3$. Then $y_1(t) = t^2, y_2(t) = t^3$, defines a parametrization of (C,0) because, $y_1(0) = 0, y_2(0) = 0$, y_1, y_2 are not both in $\mathbb{C}$, $f(y_1, y_2) = t^6 - t^6 = 0$.

Remark 1.2. Given (y_1, y_2) a parametrization of $(C_f, 0)$ at 0 we have a homomorphism of $\mathbb{C}$-algebras

$$\Psi : \mathbb{C}\{x, y\} \to \mathbb{C}\{t\}$$

defined by $\Psi(x) = y_1$, $\Psi(y) = y_2$, $\Psi(a) = a$ for all $a \in \mathbb{C}$. We have $Im\Psi \neq \mathbb{C}$ and $(f) \subset ker\Psi$.

On the other hand, if we have $\Psi : \mathbb{C}\{x, y\} \to \mathbb{C}\{t\}$ a morphism of $\mathbb{C}$-algebras with $Im\Psi \neq \mathbb{C}$ and $(f) \subset ker\Psi$ then we have a parametrization given by $(\Psi(x), \Psi(y))$.

A Puiseux parametrization is a special kind of parametrization. In order to introduce it, first we have to introduce the fractionary power series field $\mathbb{C}\langle\langle x\rangle\rangle$.

Definition 1.2.2. *We denote by* $\mathbb{C}\langle\langle x\rangle\rangle = \bigcup_{n\geq 1} \mathbb{C}((x^{1/n}))$*, where* $\mathbb{C}((x^{1/n}))$ *is the Quotient field of* $\mathbb{C}[[x^{1/n}]]$*. Its elements are of the form* $s = \sum_{i\geq d} a_i(x^{1/n})^i$*,* $d \in \mathbb{Z}$ *and if* $a_d \neq 0$*, we call ord(s)=d/n the order of s.* $\mathbb{C}\langle\langle x\rangle\rangle$ *is a field and we call it* ***the fractionary power series field****.*

An element $s \in \mathbb{C}\langle\langle x\rangle\rangle$ *is a series with exponents rational numbers with common denominators, that is* $\exists$ *n such that* $s \in \mathbb{C}((x^{1/n}))$*. Note that if m is divided by n then* $\mathbb{C}((x^{1/n})) \subset \mathbb{C}((x^{1/m}))$*.*

We now come to the definition of Puiseux series.

Definition 1.2.3. *A* ***Puiseux series*** *is an element of* $\mathbb{C}\langle\langle x\rangle\rangle$ *with order strictly positive.*

Remark 1.3. Let $f \in \mathbb{C}\{x, y\}$, and let s be a Puiseux series. Then we may replace y by s in f. If s$\in \mathbb{C}((x^{1/n}))$, $f(x, s) \in \mathbb{C}[[x^{1/n}]]$. Assume $s = \sum_k a_k x^{k/n}$ and f=$\sum_{i,j} A_{i,j} x^i y^j$ then:

$$f(x, s) = \sum_{i,j} A_{i,j} x^i \left(\sum_k a_k x^{k/n}\right)^j .$$

Thus the coefficient of $x^{d/n}$ is: $\sum_{i_1 \cdots i_j + n \cdot i = d} A_{i,j} a_{i_1} \dots a_{i_j} = B_d$.

To make this clearer let us see an example.

Example 1.2. Let f=$\sum_{i+j=1}^{3} x^i y^j$ and s=$2x^{1/2} + 3x$ then:

$$f(x, s) = 2x^{1/2} + 8x + 22x^{3/2} + 28x^2 + 68x^{5/2} + 40x^3.$$

The Puiseux series are especially important in the next case,

Definition 1.2.4. *Let* $f \in \mathbb{C}\{x, y\}$*, a* ***y-root*** *of f is a Puiseux series s such that f(x,s)=0.*

About the convergence of the y-root it is treated in the reference [3].

Our goal is to find all y-roots of a given series $f \in \mathbb{C}\{x, y\}$.

We have already said that given $s \in \mathbb{C}\langle\langle x\rangle\rangle$ there exists an $n \in \mathbb{N}$ such that $s \in \mathbb{C}((x^{1/n}))$, but $s \in \mathbb{C}((x^{1/m}))$ for many values $m \neq n$. Between all $n_k \in \mathbb{N}$ such that $s \in \mathbb{C}((x^{1/n_k}))$ exists a very important number defined as follows,

Definition 1.2.5. *The* ***order of Polydromy*** *of s is:*

$$\gamma(s) = min\{n \geq 1 \mid s \in \mathbb{C}((x^{1/n}))\}. \tag{1.1}$$

Remark 1.4.
- Let $s = \sum_{i \geq d} a_i x^{i/n}$ be an element of $\mathbb{C}\langle\langle x\rangle\rangle$ then, $n = \gamma(s)$ iff there is no a common factor for n and all the i such that $a_i \neq 0$.

- Let $s \in \mathbb{C}((x^{1/n}))$ and $\varepsilon \in \mathbb{C}$ be such that $\varepsilon^n = 1$, then we have an isomorphism

$$\begin{aligned} \sigma_\varepsilon : \mathbb{C}((x^{1/n})) &\to \mathbb{C}((x^{1/n})) \\ \sum a_i x^{i/n} &\mapsto \sum a_i \varepsilon^i x^{i/n}. \end{aligned} \tag{1.2}$$

A natural question is: since s is in $\mathbb{C}((x^{1/n}))$ for different $n \in \mathbb{N}$, do the elements in the last remark $\sigma_\varepsilon(s)$ depend on the choice of such a n?. The next lemma tells us no.

Lemma 1.2.6. *Consider $s \in \mathbb{C}\langle\langle x\rangle\rangle$ and $\sigma_\varepsilon(s)$ the function defined as in the Equation (1.2). The subset $\{\sigma_\varepsilon(s) \mid \varepsilon^n = 1\}$ does not depend on the n chosen such that $s \in \mathbb{C}((x^{1/n}))$ and has cardinal $\gamma(s)$.*

This subset plays an important role in the search of the y-roots, so we describe more precisely.

Definition 1.2.7. *Let $s \in \mathbb{C}((x^{1/n}))$ be a Puiseux series, then consider , the roots of the unit ε_i, $i = 1, \cdots, n$, and σ_{ε_i} as in the Equation (1.2). The elements of $\{\sigma_{\varepsilon_i}(s) \mid i = 1, \cdots, n\}$ are called the* ***conjugations*** *of s.*

Example 1.3. Let s=$x^{2/4} + x^{6/4}$, then $s \in \mathbb{C}((x^{1/4}))$ and $\varepsilon_1 = 1, \varepsilon_2 = -1, \varepsilon_3 = i, \varepsilon_4 = -i$. We thus get:
$\sigma_{\varepsilon_1}(s) = x^{2/4} + x^{6/4}$,
$\sigma_{\varepsilon_2}(s) = (-x^{1/4})^2 + (-x^{1/4})^6 = x^{2/4} + x^{6/4}$,
$\sigma_{\varepsilon_3}(s) = i^2 x^{2/4} + i^6 x^{6/4} = -x^{2/4} - x^{6/4}$,
$\sigma_{\varepsilon_4}(s) = (-i)^2 x^{2/4} + (-i)^6 x^{6/4} = -x^{2/4} - x^{6/4}$.

An important property when we want to calculate all the y-root of a given element $f \in \mathbb{C}\{x, y\}$.

Proposition 1.2.8. *Let $s \in \mathbb{C}((x^{1/n}))$ be a Puiseux series, and let ε_i, $i = 1, \cdots, n$, be the n roots of the unit. Consider $f \in \mathbb{C}\{x, y\}$, if s is a y-root of f then $\sigma_{\varepsilon_i}(s)$ are y-roots of f for $i = 1, \cdots, n$.*

The way to get the y-roots is through the Newton polygon.

Definition 1.2.9. ***(Newton Polygon)***
Let $f = \sum_{\alpha,\beta} A_{\alpha,\beta} x^\alpha y^\beta \in \mathbb{C}\{x, y\}$. We define the set:

$$\Delta(f) = \{(\alpha, \beta) \in \mathbb{R}^2 \mid A_{\alpha,\beta} \neq 0\}.$$

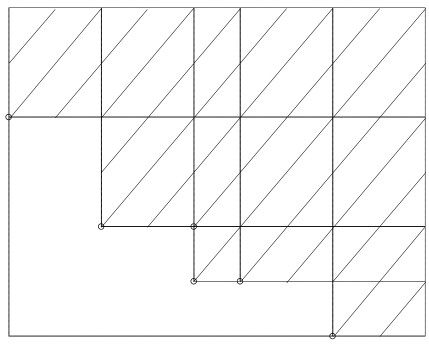


Figure 1.1: $\Delta'(f)$

Now we construct:

$$\Delta'(f) = \Delta(f) + (R_{\geq 0})^2 = \{(\alpha + r_1, \beta + r_2) \mid A_{\alpha,\beta} \neq 0, r_1 \geq 0, r_2 \geq 0\}.$$

Let $\overline{\Delta}(f)$ *be the convex hull of* $\Delta'(f)$*, i.e. the smallest convex subset of* $\mathbb{R}^2$ *containing* $\Delta'(f)$*.*

Then we consider the boundary of $\overline{\Delta}(f)$*, which turns out to be a semiline parallel to the x-axis and another semiline parallel to the y-axis and a polygonal line (may be a point) joining both semilines. This polygonal line is what we call* ***Newton polygon*** *of f, and we denote it by N(f). Its vertices are points of* $\Delta(f)$*.*

Example 1.4. Let $f = y^4 - x^2y^2 - 2x^4y^2 + x^4y + x^5y + x^7$, then:

$$\Delta(f) = \{(0,4), (2,2), (4,2), (4,1), (5,1), (7,0)\}.$$

In the Figure 1.1 we show $\Delta'(f)$ and in the Figure 1.2 the Newton polygon, that has been derived just like we have described in Definition 1.2.9.

An important case is when the Newton polygon N(f) is just a point α_0, β_0. If (α, β) is such that $A_{\alpha,\beta} \neq 0$ then $\alpha \geq \alpha_0$ and $\beta \geq \beta_0$ so:

$$f = x^{\alpha_0} y^{\beta_0} \sum_{\alpha,\beta} A_{\alpha,\beta} x^{\alpha-\alpha_0} y^{\beta-\beta_0}.$$

Then we can deduce the following proposition.

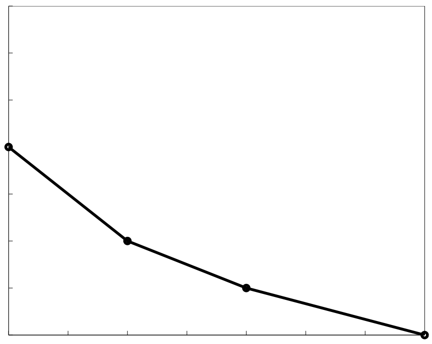


Figure 1.2: Newton polygon

Proposition 1.2.10. *Let $f \in \mathbb{C}\{x,y\} \backslash \mathbb{C}$ and N(f) its Newton polygon. N(f) is a point (α_0, β_0) iff $f = x^{\alpha_0} y^{\beta_0} g$, where $g \in \mathbb{C}\{x,y\}$ such that it is neither divided by x nor by y.*

We have seen that the structure of the Newton polygon N(f) is very important to develop some essential properties of our function f. In this sense the next definition is important.

Definition 1.2.11. *Let $f \in \mathbb{C}\{x,y\} \setminus \mathbb{C}$ and N(f) its Newton polygon. We call the biggest y-coordinate of the vertices of N(f),* ***height of N(f)****, and we denote it by* ***h(N(f))****.*

Example 1.5. Let us consider the points of the Newton polygon in the Example 1.4, with vertex $(0,4),(2,2),(4,1),(7,0)$. The y-coordinates are $4,2,1,0$ and the biggest one is 4. Thus we say that h(N(f))=4.

The height of the polygon gives us the answer to the question, whether there exists a y-root of a given $f \in \mathbb{C}$. This answer is described in the following proposition.

Proposition 1.2.12. *Let N(f) be the Newton polygon of $f \in \mathbb{C}\{x,y\} \setminus \mathbb{C}$. If h(N(f))=0, f does not have y-roots. Moreover f is of the form: f=$x^{\alpha_0} \sum_{\alpha \geq \alpha_0, \beta \geq 0}$, where $(\alpha, 0)$ is the only point in N(f).*

1.2.1 Searching for the y-roots

Now we are going to deduce an explicit formula for the y-roots, and then we will give an algorithm describing this process.

Let $f \in \mathbb{C}\{x,y\} \setminus \mathbb{C}$ and s be a Y-roots of f. We write s in the form: s=a $x^{m/n}$+(term of higher order), where $a \neq 0$ (i.e. $m/n = ord_x(s)$). We have to look for the terms of lowest possible order in f(x,s), we consider f as follows:

$$0 = f(x,s) = \sum_{\alpha,\beta} x^{\alpha}(ax^{m/n} + \ldots)^{\beta} = \sum_{\alpha,\beta} A_{\alpha,\beta} a^{\beta} x^{\alpha+\beta m/n}.$$

Thus the terms of lowest order are those with order, $\min\{\alpha + \beta m/n \mid A_{\alpha,\beta} \neq 0\}$. Let d/n be such a lowest order. The terms in f of degree d/n are:

$$\sum_{n\alpha+m\beta=d} A_{\alpha,\beta} a^{\beta} x^{(n\alpha+m\beta)/n} = (\sum_{n\alpha+m\beta=d} A_{\alpha,\beta} a^{\beta}) x^{d/n}.$$

If s=$ax^{m/n} + \ldots$ is a Y-root, then there exists a side Γ of N(f) such that its slope is -n/m (because the line nx+my=d has slope -n/m and contains at least two points of $\Delta(f)$). Let β_0 the lowest of the y-coordinates of the vertices of Γ. Associated to Γ we define the polynomial:

$$F_{\Gamma}(z) = \sum_{(\alpha,\beta)\in\Gamma} A_{\alpha,\beta} z^{\beta-\beta_0} \in \mathbb{C}[z].$$

This polynomial allow us calculate the first coefficient "a" of the y-root, only take "a" us a non zero root of F_{Γ}.

Let s=$x^{m/n}(a+s^1)$ where $s^1 = bx^{m_1/n_1}+\ldots$ then s=$ax^{m/n}+bx^{m/n+m_1/n_1}+\ldots$. What do we know about s^1?.

Consider the change $x = x_1^n$ and $y = x_1(a + y_1)$, and suppose that gcd(m,n)=1. This change makes sense because $\mathbb{C}\langle\langle x\rangle\rangle \subseteq \mathbb{C}\langle\langle x_1\rangle\rangle$, $x_1 = x^{1/n}$.

This leads to the next formula.

$$f(x,y) = f(x_1^n, x_1^m(a+y_1)) = \sum_{\alpha,\beta} A_{\alpha,\beta} x_1^{n\alpha}(x_1^m(a+y_1))^{\beta} = \sum_{\alpha,\beta} A_{\alpha,\beta} x_1^{n\alpha+m\beta}(a+y_1)^{\beta}.$$

If the side Γ is $nx + my = d$ then $\forall \alpha,\beta \quad A_{\alpha,\beta} \neq 0, n\alpha + m\beta \geq d$, then $f(x,y) = x_1^d f_1(x_1,y_1)$, $f_1 \in \mathbb{C}\{x_1,y_1\}$ where:

$$f_1(x_1,y_1) = \sum_{\alpha,\beta} x_1^{n\alpha+m\beta-d}(a+y_1)^{\beta}$$

Now we have to replace s^1 in f_1:

$$f_1(x_1,s^1) = x_1^{-d} f(x_1^n, x_1^m(a+s^1)) = x^{-d/n} f(x, x^{m/n}(a+s^1)) = x^{-d/n} f(x,s).$$

This shows, $f(x,s) = 0 \Leftrightarrow f(x_1, s^1) = 0$, i.e. s is a Y-root of f $\Leftrightarrow$ s^1 is a Y-root of f_1. Then if $s^1 \neq 0$, $s^1 = bx^{m_1/n_1} +$ (higher order), where $-n_1/m_1$ is the slope of the side Γ_1 of N(f) and b is the root of F_{Γ_1}.

Once we have calculated the first term of s^1, we substitute in s and we obtain:

$$s = x^{m/n}(a + s^1) = x^{m/n}(a + bx_1^{m_1/n_1} + \ldots) = x^{m/n}(a + bx_1^{m_1/(n \cdot n_1)} + \ldots).$$

If we iterate this process we obtain the y-root, but obviously this process may be infinity, for this reason the process only finish in few cases or should be broken.

Let show an example to clarify this process.

Example 1.6. Let $f = y^2 - x^3$. We look for the y-roots of f. 0 it is not a y-root, because f(x,0)=$-x^3 \neq 0$. The Newton polygon has only one side with vertices (3,0) and (0,2). Thus:

$$0 = f(x,s) = (ax^{m/n} + \ldots)^2 - x^3 = a^2x^{2m/n} + (higher \quad order \quad than \quad 2m/n) - x^3$$

Which is the term of lowest order:

- $a^2x^{2m/n}$, if $2m/n < 3$.
- $-x^3$, if $2m/n > 3$.
- $a^2x^{2m/n} - x^3$, if $2m/n = 3$.

Suppose that $2m/n = 3$: $(a^2 - 1)x^3 = 0 \Rightarrow a^2 - 1 = 0 \Rightarrow$ a is a root of $F_\Gamma = A_{(0,2)}z^2 + A_{(3,0)}z^0 = z^2 - 1 \Rightarrow$ a=1 or a=-1. If a=1:

$m/n = 3/2$ (because they are primes) $\Rightarrow$ m=3,n=2. We make the change $x = x_1^2$, $y = x_1^3(1 + y_1)$ then we have:

$$y^2 - x^3 = x^6(1 + y_1)^2 - x_1^6 = x_1^6[(1 + y_1)^2 - 1].$$

$f_1(x_1, y_1) = y_1^2 + 2y_1$ where the y_1-roots of f_1 are: $s^1 = 0$ or $s^1 = -2$ but -2 it is not a Puiseux root because it should be positive. Then a Y-root is:

$$s = x^{3/2}(1 + s^1) = x^{3/2}.$$

The same for a=-1: s=$-x^{3/2}$.

Once we have the Newton polygon of a given f, we have trivial criterion to know if 0 is a y-root of f.

Proposition 1.2.13. *Let $f \in \mathbb{C}\{x,y\}$ and N(f) its Newton polygon. Then,* 0 *is a Y-root of f $\Leftrightarrow$ f is divisible by y $\Leftrightarrow$ N(f) ends over the x-axis.*

In the next we show a complete description of the Newton-Puiseux algorithm to find the y-roots.

Algorithm 1.2.14. ***(Newton-Puiseux)***
input: f(x,y)$\in \mathbb{C}\{x,y\}$.
output: A y-root of f.
Step 0*:*
We choose one of the next two possibilities(always we can choose one of them):

(0-a) *Only if N(f) end over the x-axis: We make s=0 (end).*

(0-b) *Only if exist any side in N(f) (i.e. N(f) is not a point):*

Let Γ be a side of N(f) (any of them), let nx+my=d the equation the line and suppose gcd(m,n)=1 (where $-n/m$ its slope and d/m is the intersection with the y-axis) and let $a \in \mathbb{C}$ a root of $F_\Gamma \in \mathbb{C}[z]$ (this is possible because of $\mathbb{C}$ is algebraic closed). Now we make the change, $x = x_1^n$, $y = x_1(a + y_1)$ and we define $f_1(x_1, y_1) = x_1^{-d} f(x, y)$, we write $s = x^{m/n}(n + s^1)$ where s^1 is the element we have to find.

Step i -1$\rightarrow$ Step i*:*
Only we made the step i if in the last step we made (i-1,b). We have constructed $\forall j \leq i - 1$:

- *$f_j \in \mathbb{C}\{x_j, y_j\}$.*
- *Γ a side of N(f_j) with equation $n_j x + m_j y = d_j$.*
- *a_j a root of F_{Γ_j}.*

Step i*:*

(i-a) *Only if N(f_i) end over the x-axis: We make $s_i = 0$ (end).*

(i-b) *Only if exist a side in $N(f_i)$ (i.e. N(f) is not a point): we choose a side Γ_i with equation $n_i x + m_i y = d_i$ and with gcd(n_i, m_i)=1, we get a root from F_{Γ_i} and we make the change $x_i = x_{i+1}^{n_i}$, $y_i = x_{i+1}^{m_i}(a_i + y_{i+1})$ then $f_{i+1}(x_{i+1}, y_{i+1}) = x_{i+1}^{-d_{i+1}} f_i(x_i, y_i)$.*

idea of the proof:

- *If $\exists$ i such that $s^i = 0$ then in trivial.*
- *In other case, the algorithm is infinite. Is easy to prove that $ord_x(f(x,s))$ is as big as we want then $ord_x(f(x,s)) = \infty \Rightarrow f(x,s) = 0$.*

Let us go to compute completely an example following this algorithm.

Example 1.7. Calculate the y-roots of, $f(x,y) = y^4 - 2x^3y^2 - 4x^5y + x^6 - y^7$.

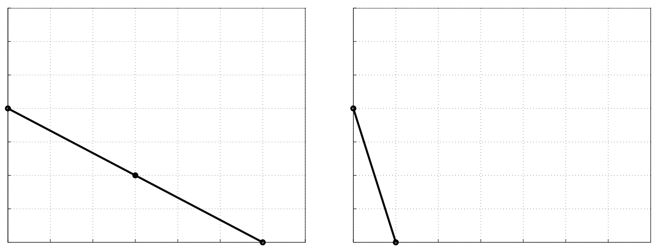


Figure 1.3: N(f) and $N(f_1)$

Step 0.

(0-a) In the polygon we only have one side (see Figure 1.3 N(f)), this side does not finish over the x-axis, then we can not apply this part.

(0-b) The side has slope=$-2/3$, the side is on the line $y = -2/3 + 4x$, then we deduce $n = 2$,$m = 3$ (for sure $gcd(2,3) = 1$) and $d = 12$. Hence we have to compute the roots of the next polynomial:

$$F_\Gamma(z) = z^4 + (-2)z^2 + z^0 = (z^2 - 1)^2.$$

The roots of $F_\Gamma(z)$ are, $a = 1$ and $a = -1$ (both are double roots).
If we choose $a = 1$ we have the y-root written of the form $s = x^{3/2}(1 + s^1)$. Now we make the change, $x = x_1^2$, $y = x_1^3(1 + y_1)$, this change turn out the

equation: $f(x,y) = x_1^{12} f_1(x_1,y_1)$ where $f_1(x_1,y_1) = y_1^4 + 4y_1^3 + 4y_1^2 - 4x_1y_1 - x_1^2 - 4x_1 + 1$.

Step1.

As before we only can make (1-b). Also here we have only one side Γ_1 in $N(f_1)$ (see Figure 1.3). The side has slope $-m/n = -2$, then we deduce $n = 2$,$m = 1$ and $d_1 = 2$. The line containing the side is $y = -2x + 2$. Hence we have to compute the roots of the next polynomial:

$$F_{\Gamma_1}(z) = 4z^2 - 4.$$

The roots of $F_{\Gamma_1}(z)$ are $a_1 = 1, a_1 = -1$. We choose $a_1 = 1$, since s and s^1 are:

$$s = x^{3/2}(1 + s^1), \qquad s^1 = x_1^{1/2}(1 + s^2) = x^{1/4}(1 + s^2).$$

Then substituting s^1 in s we get:

$$s = x^{3/2}(1 + x^{1/4}(1 + s^2)).$$

We make the change, $x_1 = x_2^2$, $y_1 = x_2(1 + y_2)$ then:

$$f_2(x_2, y_2) = x_2^{-2} f_1(x_2^2, x_2(1 + y_2)).$$

Step2.

Now we can only make the step (2-a) and we finish, then $s^2 = 0$. Finally we obtain:

$$s = x^{3/2}(1 + x^{1/4}) = x^{3/2} + x^{7/4}.$$

If s is a Y-root then all the conjugates are also y-roots. Since h(N(f))=4 (very easy to check), we have to consider the four roots of the unit, $\varepsilon_1 = 1, \varepsilon_2 = -1, \varepsilon_3 = i, \varepsilon_4 = -i$. Then the conjugates are:

- $\sigma_{\varepsilon_1}(s) = x^{6/4} + x^{7/4}$.
- $\sigma_{\varepsilon_2}(s) = (-1)^6 x^{6/4} + (-1)^7 x^{7/4} = x^{6/4} - x^{7/4}$.
- $\sigma_{\varepsilon_3}(s) = (i)^6 x^{6/4} + (i)^7 x^{7/4} = -x^{6/4} - ix^{7/4}$.
- $\sigma_{\varepsilon_4}(s) = (-i)^6 x^{6/4} + (-i)^7 x^{7/4} = -x^{6/4} + ix^{7/4}$.

We are interesting in a special kind of parametrization.

Definition 1.2.15. *A parametrization (x(t),y(t)) it is say a* ***Puiseux parametrization*** *if:*

$$x(t) = t^n, \qquad y(t) = at^m + a_{m+1}t^{m+1} + \ldots; a \neq 0.$$

We can get this parametrization with the Newton-Puiseux procedure, using the Newton polygon to f. If a y-root of f is of the form:

$$s(x) = \sum_{j \geq m} a_j x^{j/n}.$$

A Puiseux parametrization can be obtained by:

$$x = t^n, y = \sum_{j \geq m} a_j t^j.$$

Note: In the next when we speak about parametrization we will consider the Puiseux parametrization of the form:

$$x(t) = t^n, y(t) = \sum_{j \geq m} a_j t^j \tag{1.3}$$

where we can suppose $ord_t(x(t)) = n < ord_t(y(t)) = m$ only choosing x transverse.

Chapter 2

Irreducible plane curves singularities

2.1 Some topological invariants

In this section we introduce and study several topological invariants of irreducible plane curve singularity, but until the next section, in Theorem 2.2.7, we will not prove that all of them are equivalents. Also all of them are equivalent to the topological type, although this fact it is not proved in this work because is too hard . We follow the description in the book we have cited in the reference [9].

Let (C,0) be an irreducible plane curve singularity defined by $f \in \mathbb{C}\{x, y\}$, $\mathcal{O}_{\mathcal{C}} = \mathbb{C}\{x, y\}/(f)$ its local ring and $\mathfrak{m}_C$ the maximal ideal. Assume that $\mathcal{O}_{\mathcal{C}} \subset \mathbb{C}\{t\}$ is the normalization of $\mathcal{O}_{\mathcal{C}}$, in fact speak about normalization in the same as speak about parametrization. We introduce our first invariants.

Definition 2.1.1. ***(1)*** *We call* $\delta(C, 0) := dim_{\mathbb{C}}(\mathbb{C}\{t\}/\mathcal{O}_{\mathcal{C}})$ *the* ***delta-invariant of (C,0)****.*

(2) *We call* $\Gamma(C, 0) := \{ord_t(a) \mid a \in \mathcal{O}_{\mathcal{C}}, a \neq 0\}$ *the* ***semigroup of (C,0)****.*

(3) *We define* $c(C, 0) := min\{\alpha \in \Gamma(C, 0) \mid \alpha + \mathbb{N} \subset \Gamma(C, 0)\}$ *the* ***conductor*** *of (C,0), and the* ***conductor ideal*** *of (C,0) is* $\langle t^{c(C,0)} \rangle \trianglelefteq \mathbb{C}\{t\}$*.*

Another interpretation of the last invariants is given by the following remark.

Remark 2.1. • If (C,0) has a parametrization of the form $(t^n, s(t))$, then $\Gamma(C,0) = \{ord_t(g(t^n, s(t)) \mid g \in \mathcal{O}_\mathcal{C}, g \neq 0\}$.

- $\delta(C,0) = \#(\mathbb{N} \setminus \Gamma(C,0))$.

Example 2.1. Consider the plane curve singularity (C,0) given by $f = y^2 - x^3$. As we already know, (see Example (1.1)), $x(t) = t^2, y(t) = t^3$ is a parametrization of the singularity. Thus the semigroup $\Gamma(C,0) = \langle 2,3\rangle$. Because of $4 = 2+2$ and $5 = 3+2$ are in $\Gamma(C,0)$, it follows that $c(C,0) = 2$, and the unique element of $\mathbb{N}/\Gamma(C,0)$ is one. Therefore $\delta(C,0) = 1$.

The next thing we want to do is to give a complete characterization of the semigroup of plane curve singularity. For this we need the definition of an important invariant, the minimal set of generators of the semigroup.

Definition 2.1.2. ***(Generators of the semigroup)***
Let (C,0) be an irreducible plane curve singularity defined by $f \in \mathbb{C}\{x,y\}$, and let $\mathcal{O}_\mathcal{C} = \mathbb{C}\{x,y\}/(f)$ be its local ring. Consider $\Gamma(C,0)$ the semigroup of (C,0). Then we define:

$\beta_0 = min\{a \in \Gamma(C,0) \mid a \neq 0\}$,

$\beta_\nu = min\{a \in \Gamma(C,0) \mid gcd(a, \beta_0, \ldots, \beta_{\nu-1}) < gcd(\beta_0, \ldots, \beta_{\nu-1})\}$,
for $\nu \geq 1$.

By construction we obtain only finitely many β_ν, say $\beta_0, \ldots, \beta_{g'}$.
We call $G(C,0) = \{\beta_0, \ldots, \beta_{g'}\}$, the ***Generators of the semigroup****.*

We have to see that the generators of the semigroup G(C,0), generates the semigroup $\Gamma(C,0)$ and they are a topological invariant. This is done in the next lemma.

Lemma 2.1.3. *Let (C,0) be an irreducible plane curve singularity with semigroup $\Gamma(C,0)$, and let G(C,0)=$\{\beta_0, \ldots, \beta_{g'}\}$ the generators of the semigroup $\Gamma(C,0)$. Then the next properties hold:*

(1) *The set G(C,0) is the minimal set of generators of $\Gamma(C,0)$.*

(2) $gcd(\beta_0, \ldots, \beta_{g'}) = 1$.

Proof: Part (1) is proved in reference [8], and part (2) it is obvious by construction. $\square$

To compute the generators of the semigroup we need another invariant, the so-called characteristic exponents.

Definition 2.1.4. ***(Characteristic Exponents)***
Let (C,0) be an irreducible plane curve singularity, and let $x(t) = t^n$; $y(t) = \sum_{i \geq n} a_i t^i$ be a parametrization of the curve. Note that we always can choose a parametrization like this as we have seen in the Equation (1.3). We define:

$$k_0 = n,$$

$$k_\nu = min\{i \mid a_i \neq 0, gcd(i, k_0, \ldots, k_{\nu-1}) < gcd(k_0, \ldots, k_{\nu-1})\}, \text{ for } \nu \geq 1.$$

*Again by construction we obtain only finitely many k_ν, say $k_0, \ldots, k_g$. Notice that $k_0 < k_1 < \ldots < k_g$, moreover $gcd(k_0, k_1, \ldots, k_g) = 1$, follows from $dim_{\mathbb{C}}(\mathbb{C}\{t\}/\mathcal{O}_C) < \infty$. We call $E(C,0) = \{k_0, \ldots, k_g\}$, the **Characteristic exponents**.*

Example 2.2. Given the irreducible plane curve (C,0) with equation $f = y^3 - x^5$, it is easy to check that its parametrization is $x(t) = t^3, y(t) = t^5$. The invariants are:

- $\Gamma(C,0) = \{3, 5, 6, 8, 9, \ldots\}$.
- $c(C,0) = 8$.
- $\delta(C,0) = \#(\mathbb{N} \setminus \Gamma(C,0)) = \#\{1, 2, 4, 7\} = 4$.
- $k_0 = 3, k_1 = 5$.
- $\beta_0 = 3, \beta_1 = 5$.

We are interested in the relation between the generators of the semigroup an the characteristic exponents. The next theorem gives us a formula that relates both invariants.

Theorem 2.1.5. *Let (C,0) be an irreducible plane curve singularity, and let $x(t) = t^n, y(t) = \sum_{i \geq n} a_i t^i$ be a parametrization of (C,0), then the minimal set of generators of $\Gamma(C,0)$, $G(C,0) = \{\beta_0, \ldots, \beta_{g'}\}$ and the set of characteristic exponents $E(C,0) = \{k_0, \ldots, k_g\}$ satisfy the relations:*

(1) $g = g\prime$

(2) $k_i = \beta_i - \sum_{j=1}^{i-1} \left(\frac{gcd(\beta_0, \ldots, \beta_{j-1})}{gcd(\beta_0, \ldots, \beta_j)} - 1 \right) \beta_j$*;* $i = 1, \ldots, g$*.*

(3) $\beta_i = k_i + \sum_{j=1}^{i-1} \left(\frac{gcd(k_0, \ldots, k_{j-1})}{gcd(k_0, \ldots, k_{i-1})} - \frac{gcd(k_0, \ldots, k_j)}{gcd(k_0, \ldots, k_{i-1})} \right) k_j$*;* $i = 1, \ldots, g$

In particular $k_0 = \beta_0, k_1 = \beta_1$*.*

Proof : Is given in the reference [9] with all details. □

Another important invariant can be calculated from the characteristic exponents.

Definition 2.1.6. *Let (C,0) be an irreducible plane curve singularity, and* $E(C, 0) = \{k_0, \ldots, k_g\}$ *the characteristic exponents of the plane curve singularity (C,0).*

Let $N_i = \frac{e_{i-1}}{e_i}$*,* $1 \leq i \leq g$*, where* $e_i = gcd(k_0, \ldots, k_i)$*. Then there exist natural numbers* $m_1, \cdots, m_g$*, such that:*

$$\frac{k_1}{n} = \frac{m_1}{N_1} \quad ; \quad \frac{k_2}{n} = \frac{m_2}{N_1 N_2} \quad ; \quad \ldots \quad ; \quad \frac{k_g}{n} = \frac{m_g}{N_1 \ldots N_g};$$

Where $n = N_1 \cdots N_g$*,* $N_i > 1$ *and* $gcd(m_i, N_i) = 1$*,* $1 \leq i \leq g$*. The pairs* $\{(m_i, N_i) \mid i = 1, \ldots, g\}$ *are called the* ***Puiseux Pairs*** *of the plane curve singularity (C,0).*

2.2 Resolution of irreducible plane curves singularities

In the following we introduce our main tool in the resolution of singularities, the *blowing-up* in the point 0 of $\mathbb{C}^2$. This is a process of a purely local nature, and the essence of the process consists in replacing the point 0 by the set of all lines through 0, i.e. by the one-dimensional projective space.

Definition 2.2.1. *We define* $Bl_0\mathbb{C}^2$ *to be the closed complex subspace*

$$Bl_0\mathbb{C}^2 = \{(p, L) \mid p \in L\} \subset \mathbb{C}^2 \times \mathbb{P}^1_{\mathbb{C}},$$

where p is a point and L is a line in $\mathbb{C}^2$*.*

There is a natural projection

$$\pi : Bl_0\mathbb{C}^2 \longrightarrow \mathbb{C}^2$$

$$(p,l) \longrightarrow p$$

which is called the ***blowing-up*** *of* $0 \in \mathbb{C}^2$*.*

We write $E := \pi^{-1}(0) \subset Bl_0\mathbb{C}^2$ *and we call it the* ***exceptional divisor*** *of* π*. Frequently, E is also called the first infinitesimal neighborhood of* $0 \in \mathbb{C}^2$*.*

Note that each point of E corresponds to a unique line in $\mathbb{C}^2$*. Moreover, the preimages of any two lines* $L \neq L' \subset \mathbb{C}^2$ *through 0 do not have any point in common. Hence, blowing up* $0 \in \mathbb{C}^2$ *separates lines through the origin.*

We can describe $Bl_0\mathbb{C}^2$ in coordinates, as we show in the remark.

Remark 2.2. $Bl_0\mathbb{C}^2 = \{(x,y,u:v) \mid xv = yu\} \subset \mathbb{C}^2 \times \mathbb{P}^1_{\mathbb{C}}$. This holds since:

$$(x,y) \in (u,v) \Leftrightarrow (x,y) = \lambda(u,v) \Leftrightarrow \exists\lambda : \ x = \lambda u, y = \lambda v \Leftrightarrow xv = yu.$$

Since $\mathbb{P}^1_{\mathbb{C}} = U_1 \cup U_2$ where $U_1 = \{(1:v) \in \mathbb{P}^1_{\mathbb{C}}\}$, $U_2 = \{(u:1) \in \mathbb{P}^1_{\mathbb{C}}\}$ are the two chart of $\mathbb{P}^1_{\mathbb{C}}$ then:

$$V_1 = Bl_0\mathbb{C}^2 \cap (\mathbb{C}^2 \times U_1) = \{(x, xv, 1:v) \mid x, v \in \mathbb{C}^2\} \cong \mathbb{C}^2,$$

$$V_2 = Bl_0\mathbb{C}^2 \cap (\mathbb{C}^2 \times U_2) = \{(yu, y, u:1) \mid y, u \in \mathbb{C}^2\} \cong \mathbb{C}^2.$$

Example 2.3. Consider the irreducible plane curve singularity (C,0) given by the equation $f = y^2 - x^3 \in \mathbb{C}\{x,y\}$. Consider the blowing-up in the chart $u = 1$, hence $y = xv$ and the equation of $\pi^{-1}((C,0))$ in the new variables x,v is:

$$(xv)^2 - x^3 = x^2(v^2 - x) = 0.$$

The exceptional divisor is $E \equiv \{x = 0\}$ and the closure of $\pi^{-1}(C\backslash 0)$ has equation $f_1 = v^2 - x$, that it is smooth but intersects E with multiplicity 2, i.e. not transversally, (see Figure 2.1).

In the following we study the effect of the blowing-up $\pi : Bl_0\mathbb{C}^2 \longrightarrow \mathbb{C}^2$ on an irreducible curve singularity $(C,0) \subset (\mathbb{C}^2, 0)$.

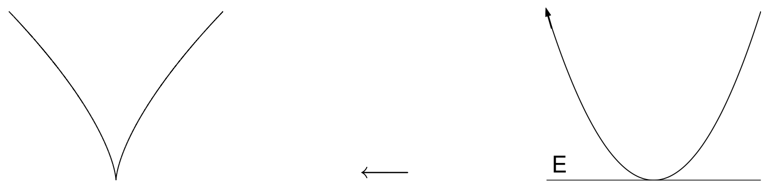


Figure 2.1: Blowing-up of the cusp

Definition 2.2.2. *Let (C,0) be an irreducible plane curve singularity and $C \subset U$ be a representative. We call $\pi^{-1}(C)$* ***the total transform*** *of C, and the closure $C^{(1)}$ of $\pi^{-1}(C \backslash 0)$ the* ***strict transform*** *of C. By abuse of notation, we denote by $C^{(1)}$ the plane curve singularity ($C^{(1)}$,p), where p is the intersection point of $C^{(1)}$ with the exceptional divisor.*

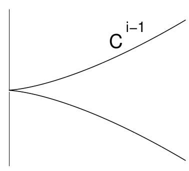
For our purpose, classifying the irreducible plane singularities, we need the desingularization of such a curves. Let us see what this means.

Definition 2.2.3. *Let (C,0) be an irreducible curve singularity and $C \subset U$ be a representative. Consider a sequence of blowing-ups.*

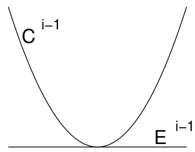
$$\begin{array}{ccccccccc} X_k & \xrightarrow{\pi_k} & X_{k-1} & \longrightarrow & \dots & \xrightarrow{\pi_2} & X_1 & \xrightarrow{\pi_1} & U \\ \cup & & \cup & & & & \cup & & \cup \\ C^{(k)} & \longrightarrow & C^{(k-1)} & \longrightarrow & \dots & \longrightarrow & C^{(1)} & \longrightarrow & C \end{array}$$

Then we define $E^{(i)} = (\pi_1 \circ \ldots \circ \pi_i)^{-1}(0)$ and the i-th strict transform $C^{(i)}$ by the closure of $(\pi_1 \circ \ldots \circ \pi_i)^{-1}(C \setminus 0)$. The above sequence of blowing-ups is called the ***standard resolution*** *if either (C,0) is smooth and $k = 0$ or for $i = 2, \ldots, k$, either:*

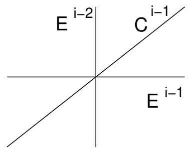
(1) *$C^{(i-1)} \subset X_{i-1}$ has one singular point, and π_i is the blowing-up of X_{i-1} in this point, or*



(2) $C^{(i-1)}$ *is smooth, but the intersection with* $E^{(i-1)}$ *in* X_{i-1} *is not transversal, or*



(3) $C^{(i-1)}$ *is smooth, intersect* $E^{(i-1)}$*transversal, but does intersect more than one component of* $E^{(i-1)}$*,*



and moreover, $C^{(k)}$ *is smooth, intersects* $E^{(k)}$ *transversally and intersects only one component of* $E^{(k)}$*.*

Of course the notion of standard resolution is only interesting since it is possible to find it for any plane curve singularity (C,0).

Theorem 2.2.4. *Let (C,0) be an irreducible plane curve singularity. Then a standard resolution of (C,0) exists.*

Proof: See reference [11], or also in reference [9]. □

After performing the standard resolution of an irreducible plane curve singularity, we get an important invariant, just looking at the multiplicities in the intersection points between $C^{(i)}$ and $E^{(i)}$.

Definition 2.2.5. *Let (C,0) be an irreducible plane curve singularity. Consider the standard resolution of (C,0)*$\subset U$ *given by,*

$$X_k \xrightarrow{\pi_k} X_{k-1} \longrightarrow \dots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} U,$$

and $(C^{(i)}, 0)$ *the i-th strict transform of (C,0).*

For $i = 1, \dots, k-1$ *let* m_i *be the multiplicity of* $C^{(i)}$ *at the intersection point with* $E^{(i)}$*, and* $m_0 = m$ *the multiplicity of* $(C, 0)$*. Then* $(m_0, \dots, m_{k-1})$ *is called the* ***multiplicity sequence*** *of (C,0).*

The last invariant in this chapter but the most important for our purpose is the graph of the standard resolution.

Definition 2.2.6. ***(Resolution graph)***
Let

$$\pi : X_k \xrightarrow{\pi_k} X_{k-1} \longrightarrow \dots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} U$$

be a standard resolution of the irreducible curve singularity $(C, 0) \subset U$*. Let* $\pi^{-1}(0) = E = \bigcup_{i=1}^{k} E_i$ *be the exceptional divisor, where* E_i *is the irreducible component of E, which comes from the i-th step in the blowing up, and* $C^{(k)}$ *the strict transform of C. To this resolution we can associate a weighted graph, the* ***resolution graph****.*

The vertices of the graph are the exceptional lines E_i *and the curve* $C^{(k)}$*. We connect two vertices* E_i, E_j *if* $E_i \cap E_j \neq \emptyset$ *and similarly , we connect a vertex* E_i *to* $C^{(k)}$ *if* $E_i \cap C^{(k)} \neq \emptyset$ *(i.e. connect* E_k *with* $C^{(k)}$*). The weight of* E_i *in the graph is i.*

We will draw " $\bullet$ *" for each* E_i *and "* $\star$ *" for* $C^{(k)}$*.*

Example 2.4. Let (C,0)$\subset$ $(\mathbb{C}^2, 0)$ be the irreducible plane curve singularity defined by $y^2 - x^5 = 0$. This is a singularity of multiplicity $m_0 = 2$.

The standard resolution consists on the next steps:

(1) We blow up, and look in the chart where we replace (x, y) by (x, xy). We get as total transform

$$(xy)^2 - x^5 = x^2(y^2 - x^3) = 0.$$

The exceptional divisor E_1 is given by $x = 0$. We pull out the factor x^2, and so the strict transform $C^{(1)}$ is given by

$$(y^2 - x^3) = 0. \tag{2.1}$$

Thus $mult(C^{(1)}) = m_1 = 2$.

(2) We look at the second blowing-up, replacing (x, y) by (x, xy) in (2.1) We get as total transform

$$(xy)^2 - x^3 = x^2(y^2 - x) = 0.$$

The exceptional curve E_2 is given by $x = 0$. Pulling out x^2 gives $C^{(2)}$

$$(y^2 - x) = 0. \tag{2.2}$$

We see that the intersection between $C^{(2)}$ and the exceptional divisor it is not transversal, moreover $mult(C^{(2)}) = m_2 = 1$.

(3) Now we change the coordinates (x, y) by (xy, y) in (2.2). Hence the total transform is

$$(y)^2 - (xy) = y(y - x) = 0.$$

The exceptional curve E_3 is given by $y = 0$. Pulling out y gives $C^{(3)}$

$$(y - x) = 0. \tag{2.3}$$

We see that C^3 intersects E_3 and E_2, and $mult(C^{(3)}) = m_3 = 1$.

(4) Now our curve $C^{(3)}$ is smooth but it intersects with more than one component of the exceptional divisor, so we make our last blowing-up changing (x, y) by (x, xy) in (2.2).

$$(xy) - x = x(y - 1) = 0.$$

The exceptional curve E_4 is given by $x = 0$. Pulling out x gives $C^{(4)}$

$$(y - 1) = 0. \tag{2.4}$$

The intersection point between $C^{(4)}$ and E_4 is (0,1), but we want this point to be (0,0), so doing a coordinate change by replacing $(x, y) \longrightarrow (x, y + 1)$, we have that $C^{(4)}$ in the new coordinates is,

$$y = 0. \tag{2.5}$$

We see that $mult(C^{(4)}) = m_4 = 1$.
Thus we get multiplicity sequence $(2, 2, 1, 1)$.

We can see in the Figure 2.2 the process we have described above, applied to our example $y^2 - x^5 = 0$.

And also we get the resolution graph as we have described in the Definition (2.2.6), it looks like as follows:

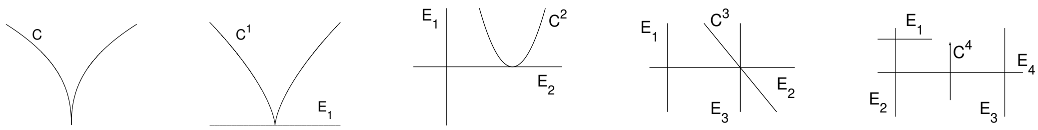
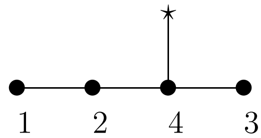


Figure 2.2: Standard resolution of $y^2 - x^5 = 0$



Along this chapter we have seen invariants that characterize the topological type of an irreducible plane curve singularity. Now we have a theorem to see that all of them are equivalent.

Theorem 2.2.7. *For a given irreducible plane curve singularity (C,0) the following data determinate each other:*

(1) *The topological type of* $(C, 0)$.

(2) *The Puiseux pairs* $P(C, 0)$.

(3) *The characteristic exponents* $E(C, 0)$.

(4) *The generators of the semigroup* $G(C, 0)$.

(5) *The multiplicity sequence* $(m_0, \ldots m_{k-1})$ *of (C,0).*

(6) *The resolution graph of (C,0).*

In particular, they are all topological invariants of (C,0).

Proof:
(1) $\Leftrightarrow$ (3) is proved in reference [3].
We have seen (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4).
(3) $\Leftrightarrow$ (5): To see this let us quote the next theorem.

Theorem 2.2.8. *Let (C,0) be an irreducible plane curve singularity with characteristic exponents* $k_0, \ldots, k_g$. *Then the multiplicity sequence can be computed by the following algorithm.*

- *If* $g = 0$ *then the multiplicity sequence is empty.*

- *If $g > 0$ we can perform the Euclidean algorithm for $\mathcal{X}_1 = k_1$, $r_{1,1} = k_0$.*

$$\mathcal{X}_1 = \mu_{1,1} r_{1,1} + r_{1,2},$$

$$r_{1,1} = \mu_{1,2} r_{1,2} + r_{1,3},$$

$$\vdots$$

$$r_{1,w(1)-1} = \mu_{1,w(1)} r_{1,w(1)}.$$

For $i = 2, \dots, g$, let $\mathcal{X}_i = k_i - k_{i-1}$ and define inductively $r_{i,1} = r_{i-1,w(i-1)}$, we perform the Euclidean algorithm with, $\mathcal{X}_i$ and $r_{i,1}$ and obtain:

$$\mathcal{X}_i = \mu_{i,1} r_{i,1} + r_{i,2},$$

$$r_{i,1} = \mu_{i,2} r_{i,2} + r_{i,3},$$

$$\vdots$$

$$r_{i,w(i)-1} = \mu_{i,w(i)} r_{i,w(i)}.$$

Then the multiplicity sequence of (C,0) contains:

$\mu_{i,j}$ *times the multiplicity* $r_{i,j}$.

In particular the characteristic exponents determine the multiplicity sequence. On the other hand, reversing the procedure allows us to compute the $k_0, \dots, k_g$ inductively from the multiplicity sequence.

Proof:See reference [9] or in reference [3]. □

Example 2.5. Let (C,0) be an irreducible plane curve singularity with characteristic exponents $k_0 = 16, k_1 = 92, k_2 = 106, k_3 = 107$. We perform the first Euclidean algorithm with, k_1 and k_0.

$$92 = 5 \cdot 16 + 12,$$

$$16 = 1 \cdot 12 + 4,$$

$$12 = 3 \cdot 4.$$

We perform the second Euclidean algorithm with, $k_2 - k_1 = 14$ and $4 = r_{1,3} = r_{2,1}$.

$$14 = 3 \cdot 4 + 2,$$

$$4 = 2 \cdot 2.$$

We perform the third Euclidean algorithm with, $k_3 - k_2 = 1$ and $2 = r_{2,2} = r_{3,1}$.

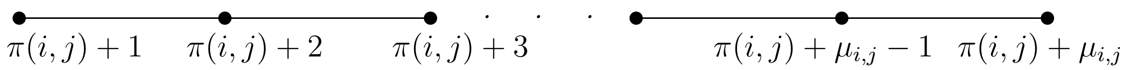
$$1 = 0 \cdot 2 + 1,$$

$$2 = 2 \cdot 1.$$

So we get the multiplicity sequence $(16, 16, 16, 16, 16, 12, 4, 4, 4, 4, 4, 4, 2, 2, 1, 1)$.

(3) $\Leftrightarrow$ (6): For this we refer to the next theorem.

Theorem 2.2.9. *Let (C,0) be an irreducible plane curve singularity, and let $k_0, \ldots, k_g$ be the characteristic exponents of (C,0). Let $\mu_{i,j}$ be determined by the g Euclidean algorithms of the previous theorem. Here $1 \leq i \leq g$ and $1 \leq j \leq w(i)$. The resolution graph consist of g Puiseux chains P_i for, $i = 1, \ldots, g$. The chains are constructed from $w(i)$ building blocks S_i^j.*



with

$$\begin{aligned}
&\pi(1,1) = 0 \\
&\pi(i,1) = \pi(i-1, w(i-1)) + \mu_{i-1,w(i-1)} \quad , \quad \textit{for } i = 2, \ldots, g. \\
&\pi(i,j) = \pi(i,j-1) + \mu_{i,j-1} \quad , \quad \textit{for } 2 \leq j \leq w(i) \textit{ and } i = 1, \ldots, g.
\end{aligned}$$

The point with the smallest weight(respectively the largest weight) in the building block S_i^j is called, the start point (respectively the end point) of the graph S_i^j.

The Puiseux chain P_i is constructed by connecting the end point of S_i^k with the start point of S_i^{k+2} as long as it is possible, that is, for $k \leq w(i) - 2$. Furthermore the end point of $S_i^{w(i)-1}$ is connected to the point of $S_i^{w(i)}$. This ends the construction of the Puiseux chain P_i.

The Puiseux chains are put into one graph by connecting the end point of $S_i^{w(i)}$ with:

(1) *the start point of S_{i+1}^1 if $\mu_{i+1,1} \neq 0$, or*

(2) *the start point of* S^3_{i+1} *if* $\mu_{i+1,1} = 0$ *and* $w(i+1) \geq 3$*, or*

(3) *the end point of* S^2_{i+1} *if* $\mu_{i+1,1} = 0$ *and* $w(i+1) = 2$.

(4) *Finally, we connect the end point of* $S_g^{w(g)}$ *with the* $\star$.

Because $w(i) \geq 2$*, the end point of* $S_i^{w(i)}$ *is, therefore, connected with three other points, and it is called* ***a contact point*** *of the graph. We have* g *contact points, and we draw the graph in such a way that each contact point creates a new level.*

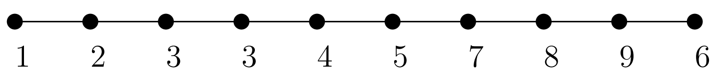
Proof: See reference [3]. □

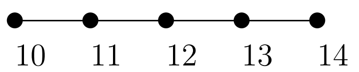
Example 2.6. We consider as in the Example (2.5), an irreducible plane curve singularity (C,0) with characteristic exponents $k_0 = 16, k_1 = 92, k_2 = 106, k_3 = 107$, we get the next table of information.

$g = 3$			
w(1)=3	S_1^1={1,2,3,4,5} $\mu_{1,1} = 5$ $r_{1,1} = 16$	S_1^2={6} $\mu_{1,2} = 1$ $r_{1,2} = 12$	S_1^3={7,8,9} $\mu_{1,3} = 3$ $r_{1,1} = 4$
w(2)=2	S_2^1={10,11,12} $\mu_{2,1} = 3$ $r_{2,1} = 4$	S_2^2={13,14} $\mu_{2,2} = 2$ $r_{2,2} = 2$	
w(3)=2	S_3^1= $\emptyset$ $\mu_{3,1} = 0$ $r_{3,1} = 2$	S_3^2={15,16} $\mu_{3,2} = 2$ $r_{3,2} = 1$	

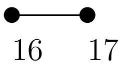
With this table we have all the information to construct the resolution graph.

First we construct each block P_i.

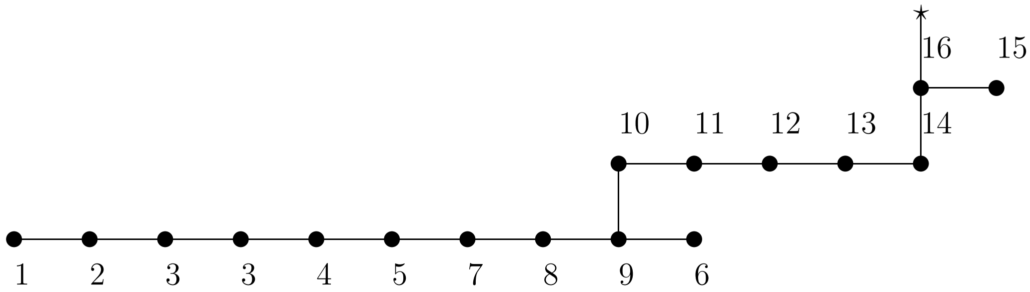
P_1 :


P_2 :


P_3 :



And the whole graph is,



Chapter 3

Reducible plane curves singularities

In the following we want to study the generalization of the last chapter, i.e. a plane curve singularity (C,0) with several branches. For this we want to describe the most important topological invariants, and the relations between them.

3.1 Classical topological invariants

Let $(C,0)$ be a plane curve singularity with equation $f = \prod_{i=1}^{r} f_i^{n_i}$ with $f \in \mathbb{C}\{x,y\}$. We are going to consider only plane curves singularities (C,0) such that $n_i = 1$, for $i = 1, \dots, r$, those curves are called reduced plane curves singularities.

With the above assumption, let $C = \bigcup_{i=1}^{r} C_i$ be a representation as the union of irreducible components, where f_i is the equation for the i-th branch $(C_i, 0)$ of (C,0). The local ring of each branch $\mathcal{O}_{C_i} = \mathbb{C}\{x,y\}/(f_i)$ is an integral domain, its integral closure $\overline{\mathcal{O}_{C_i}}$ in its quotient field is a discrete valuation ring. If $t_i \in \overline{\mathcal{O}_{C_i}}$ is a uniform parameter for the discrete valuation v_i associated to $\overline{\mathcal{O}_{C_i}}$, then $\overline{\mathcal{O}_{C_i}} = \mathbb{C}\{t_i\}$ and $v_i(g) = ord_{t_i}(g \circ f)$ for every $g \in \overline{\mathcal{O}_{C_i}}$.

We start with the first invariant, the so called semigroup of the curve.

Definition 3.1.1. ***The semigroup*** $\Gamma = \Gamma(C,0)$ *of the curve* $C = \bigcup_{i=1}^{r} C_i$ *is the subsemigroup of* $\mathbb{Z}^r_{\geq 0}$ *which consists of the elements of the form* $v(g) = (v_1(g), \dots, v_r(g))$ *for* $g \in \mathcal{O}_{\mathbb{C}^2,0} = \mathbb{C}\{x,y\}$ *with* $v_i(g) < \infty$ *for all* $i = 1, \dots, r$.

Remark 3.1. A description of the semigroup of a reducible plane curve singularity is not so simple. The main reason consists of the fact that the semigroup of a reducible plane curve singularity is not finitely generated. For example, the semigroup Γ of the plane curve singularity curve $(C,0) \equiv \{xy = 0\}$ consists of all elements of the form (v_1, v_2) with either $v_1 = v_2 = 0$, or $v_1 \geq 1, v_2 \geq 1$. Thus its minimal set generators consists of 0 and the elements $(v,1)$ and $(1,v)$ with $v \geq 1$.

Another important invariant, but not so important as the last one, is defined in the next.

Definition 3.1.2. *Let (C,0) be a reduced plane curve singularity, defined by $f = \prod_{i=1}^{r} f_i$, where f_i is the equation of the branch $(C_i, 0)$.*

We will denote by δ ***the conductor*** *of Γ, that is, the minimal element δ of Γ such that $\delta + \mathbb{N}^r \subseteq \Gamma$. The coordinates of δ are $\delta_i = c_i + \sum_{i \neq j} ord(f_i \cdot f_j)$, $i = 1, \ldots, r$, c_i being the conductor for the semigroup $\Gamma_i \subseteq \mathbb{N}$ corresponding to the branch f_i.*

The most useful invariants in the reducible case are, the system of multiplicity sequences and the resolution graph. We begin with the definition of the system of multiplicity sequences. This is already defined for the irreducible case (see Definition 2.2.5). With reducible curves we must proceed somewhat more careful since we must describe, among other things, how long the preimages of the individual components stay together, and when they separate during the blowing up process.

As in the irreducible case we first of all need a generalization of the concept of standard resolution for reduced plane curve singularity. Let us show what we mean with this.

Definition 3.1.3. *Let $C = \bigcup_{i=1}^{r} C_i \subset U$ be a small representative of a reducible reduced plane curve singularity (C,0), with branches $(C_1, 0), \ldots, (C_r, 0)$, $r \geq 2$. We consider the sequence of mappings*

$$X_N \xrightarrow{\pi_N} X_{N-1} \xrightarrow{\pi_{N-1}} \ldots \xrightarrow{\pi_{i+1}} X_i \xrightarrow{\pi_i} \ldots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} U$$

defined recursively as follows. Suppose $\pi_1, \ldots, \pi_i$ have already been defined. Denote by $E^{(i)} = (\pi_1 \circ \ldots \circ \pi_i)^{-1}(0)$ the exceptional divisor in X_i, by $E_j^{(i)}$ the irreducible components of $E^{(i)}$ although we will write only E_j if there is no confusion, and by $C^{(i)} = \overline{(\pi_1 \circ \ldots \circ \pi_i)^{-1}(C \setminus 0)}$ the strict transform of C in

X_i.
Then let

$$X_{i+1} \xrightarrow{\pi_{i+1}} X_i$$

be the blowing-up of X_i in all points of $C^{(i)} \cap E^{(i)}$ such that, are still singular on $C^{(i)}$, or are a nontransversal intersection of $C^{(i)}$ with $E^{(i)}$, or $C^{(i)}$ intersect $E^{(i)}$ in more than one component.
Then

$$X_k \xrightarrow{\pi_k} X_{k-1} \xrightarrow{\pi_{k-1}} \dots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} U$$

is called the ***standard resolution*** *of (C,0) if k is the minimal number such blowing-up processes π_i such that, all branches of $C^{(k)}$ are smooth, do not intersect each other, do intersect just one component of $E^{(k)}$, and do intersect this component transversally.*

Example 3.1. Consider two smooth curves (C,0) and (D,0) given by equations $y = 0$ and $y = x^k$ respectively.

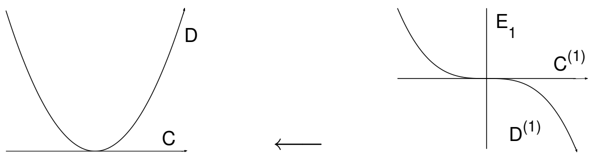


Figure 3.1: The first blowing up

Those curves have the same tangents, so we do not expect that the strict transform separates after one blowing-up. We blow up, and look at the chart $u = 1$, so $y = xv$. The strict transform of (C,0) is given by $v = 0$. The total transform of (D,0) has equation $xv - x^k = of0$. The exceptional divisor E_1 has equation $x = 0$, so that $D^{(1)}$ has equation $v = x^{k-1}$. We get the Figure 3.1.

We can continue blowing-up. After k blowing-ups we get the Figure 3.2. Now if we have two smooth curves (C,0) and (D,0) we always can choose local coordinates such that (C,0) is given by $y = 0$, and (D,0) is given by $y = x^k$. So we see from this example, that after one blowing-up the intersection multiplicities of the strict transforms has decreased by one. In particular, we see that after finitely many blowing-ups, the strict transforms of (C,0) and (D,0) become separated.

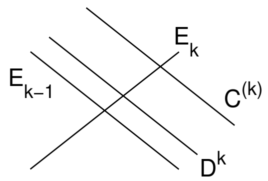


Figure 3.2: After k blowing-ups

Now a natural question arises: is the construction of the standard resolution always possible. Let us see the next theorem to answer this question.

Theorem 3.1.4. *Every germ of a plane curve singularity has a standard resolution.*

proof: We have to prove that the process of constructing $\pi_{i+1} : X_{i+1} \longrightarrow X_i$ as described above stops after a while, that is, there exists an n such that $C^{(n)}$ has no singularities on $C^{(n)} \cap E^{(n)}$, intersect $E^{(n)}$ transversally and does not intersect two components of $E^{(n)}$. From the irreducible case, we know that we can resolve the singularities of all branches such that after a while we have at most nontransversal intersections of $C^{(n)}$ and $E^{(n)}$. Now it might still happen that two (smooth) branches of $C^{(n)}$ intersect each other. But we know that the intersection multiplicity of two smooth curves drops with each blowing-up, see the Example 3.1. This proves the theorem. $\square$

As a consequence of the last definition it is possible to define the next elements.

Definition 3.1.5. *Let (C,0) be a reduced plane curve singularity, and $(C_j, 0)$ its irreducibles branches for $j = 1, \cdots, r$. Consider a standard resolution,*

$$X_k \xrightarrow{\pi_k} X_{k-1} \xrightarrow{\pi_{k-1}} \dots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} U,$$

where $C_j^{(i)}$ the strict transform of the curve $(C_j, 0)$ in X_i, i.e. $C^{(i)} = \cup_{j=1}^r C_j^{(i)}$, and $E^{(i)}$ the exceptional divisor in X_i. We define:

- *The points in $C^{(i)} \cap E^{(i)}$ are the **infinitely near points** of the singular point in X_i.*
- *Consider $j \in \mathbb{N}$ such that, for some i and k assume that $C_i^{(j)} \cap C_k^{(j)} \neq \emptyset$, but $C_i^{(j+1)} \cap C_k^{(j+1)} = \emptyset$. Then $\gamma(C_i, C_k) := j + 1$ is called the **contact number** of the branches $(C_i, 0)$ and $(C_k, 0)$.*

Now we are ready to define one of the most important invariants in the reducible case.

Definition 3.1.6. *Let $(C, 0)$ be a reduced plane curve singularity, and let*

$$X_k \xrightarrow{\pi_k} X_{k-1} \xrightarrow{\pi_{k-1}} \dots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} U$$

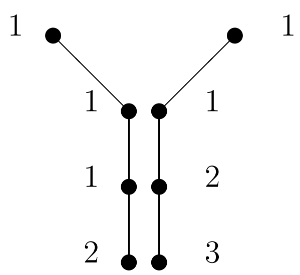
be the standard resolution of $(C, 0)$. Now assume that $(C, 0)$ decompose into the irreducible components $(C_1, 0), \dots, (C_r, 0)$, consider $C_j^{(i)}$ the strict transform of the curve $(C_j, 0)$, we denote by $m_{j,i} := mult(C_j^{(i)}, P_j^{(i)})$ for $i = 0, \dots, k$, where $P_j^{(i)}$ is the unique intersection point between $(C_j^{(i)}, 0)$ and $E^{(i)}$. Then the ***system of multiplicity sequence*** *of (C,0) is given by the extended multiplicity sequences,*

$$(m_{j,0}, m_{j,1}, \dots, m_{j,k}) \ , \ j = 1, \dots, r \ ,$$

of the branches, together with partitions $\mathcal{P}_i$ of the sets $\{(1, i), \dots, (r, i)\}$, $i \geq 0$ defined as follows: (j,i) and (l,i) belong to the same subset if and only if the strict transform of $(C_j, 0)$ and $(C_l, 0)$ intersect in E_i.

The system of multiplicity sequences of reduced plane curve singularity can be illustrated in the diagram as shown in the following.

Example 3.2. Let $f = (x^2 - y^3)(x^3 - y^5)$. Then the multiplicity sequence of the two branches are (2,1,1), resp. (3,2,1,1) and the system of multiplicity sequences can be illustrated in the diagram,



This scheme contains the same information as the system of multiplicity sequences. Here, we indicated if one branch intersects to each others by close proximity of the corresponding points, according with this $\mathcal{P}_0 = \{(0, 1), (0, 2)\}, \mathcal{P}_1 = \{(1, 1), (1, 2)\}, \mathcal{P}_2 = \{(2, 1), (2, 2)\}, \mathcal{P}_3 = \{\{(3, 1)\}, \{(3, 2)\}\}$.

The other important invariant in the reducible case is the resolution graph, and it is again a generalization of the irreducible case, more precisely we define it as follows:

Definition 3.1.7. *Let (C,0) be a reduced plane curve singularity defined by* $f = \prod_{i=1}^{r} f_i \in \mathbb{C}\{x, y\}$*. We shall assume (C,0) with r irreducible components* $(C_i, 0)$*. Let*

$$X_k \xrightarrow{\pi_k} X_{k-1} \xrightarrow{\pi_{k-1}} \dots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} U,$$

be the standard resolution of (C,0), with the notation from Definition 3.1.3 we consider associated with π *one can construct the* ***resolution graph****,* $A(f)$*, as the dual figure of E. More precisely:*

(1) *Each irreducible component of E is represented by a point in A(f).*

(2) *Two points in A(f) are joined if and only if their corresponding components in E meet.*

(3) *For each irreducible component* $C_j^{(k)}$ *of* $C^{(k)}$*, we draw a star* $\star$ *joined to the point corresponding to the only component of E meeting* $C_j^{(k)}$*.*

(4) *The points of the type (1) are weighted. Denote by E(P) the component of E corresponding to the point P of A(f), and let w(P) be the number of blowing-ups needed to build the divisor E(P). Then the weight of P is w(P).*

Example 3.3. We consider the reduced plane curve singularity $(C, 0)$ defined by the equation $f = x(x^2 - y^3)(x^3 - y^2)$, where $(C_1, 0)$,$(C_2, 0)$ and $(C_3, 0)$ are the branches of $(C, 0)$ defined by x, $x^2 - y^3$ and $x^3 - y^2$ respectively, then we have the following invariants:

- Standard resolution

  The interesting part of the strict transform $C_1^{(1)}$ and $C_2^{(1)}$ of the blowing-up of the branches $(C_1, 0)$ and $(C_2, 0)$ in the second chart are:

  $C_1^{(1)}$: defined by $x = 0$.
  $C_2^{(1)}$: defined by $x^2 - y = 0$.
  E_1: defined by $y = 0$.

In the first chart we obtain:
$C_3^{(1)}$: defined by $x - y^2 = 0$.
E_1: defined by $x = 0$.

So we have the Figure 3.3

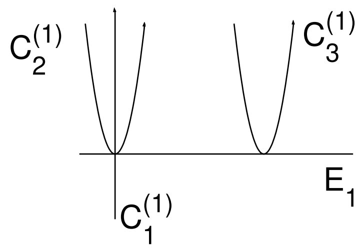


Figure 3.3: first blowing-up

We go on blowing-up in the corresponding points $P_1^{(1)} = C_2^{(1)} \cap E_1$ and $P_2^{(1)} = C_3^{(1)} \cap E_1$, where we have:

In $P_1^{(1)}$
$C_1^{(2)}$: defined by $x = 0$.
$C_2^{(2)}$: defined by $x - y = 0$.
E_2: defined by $x = 0$.

In $P_2^{(1)}$
$C_3^{(2)}$: defined by $x - y = 0$.
E_2: defined by $y = 0$.
See Figure 3.4.

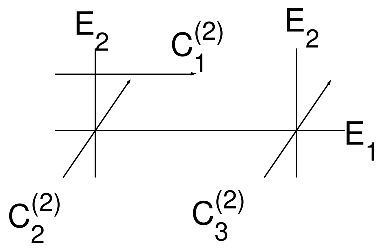


Figure 3.4: second blowing-up

And finally we blow up in $P_1^{(2)} = C_2^{(2)} \cap E_2$ and $P_2^{(2)} = C_3^{(2)} \cap E_2$, where we have:
In $P_1^{(2)}$
$C_2^{(3)}$: defined by $1 - y = 0$.
E_3: defined by $x = 0$.
In $P_2^{(2)}$
$C_3^{(3)}$: defined by $1 - y = 0$.
E_2: defined by $x = 0$.
See Figure 3.5.

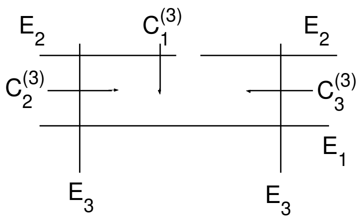
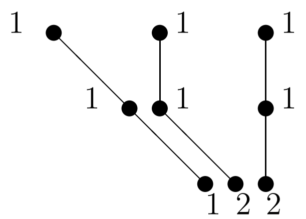
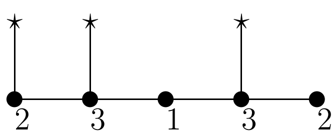


Figure 3.5: third blowing-up

- Multiplicity sequence.



- Resolution graph



3.1.1 Relations between the invariants

Having introduced the most important invariants, it would be very interesting if all of them determine each others. This is something we are going to do in the next theorems. We start with the relations between the resolution graph of a curve, and the resolution graphs of the branches.

Theorem 3.1.8. *Let* $(C,0) = \bigcup_{i=1}^{r}(C_i,0)$ *be a reduced plane curve singularity. The following data determine each other:*

(1) *The resolution graph of (C,0).*

(2) *The resolution graphs of the branches* $(C_i,0)$*, together with the contact numbers* $\gamma(C_i,C_j)$ *for* $i \neq j$*.*

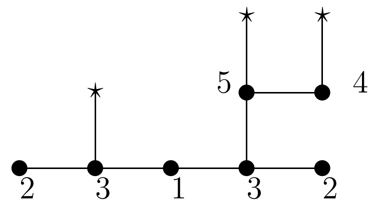
proof:

Since the proof of the theorem contains the algorithm which we are going to use for calculating the resolution graph, we rephrase it here for the convenience of the reader. It can be found in the reference [9].

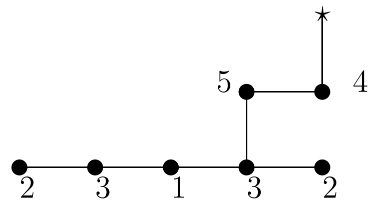
(1)⇒(2) First suppose we know the resolution graph of (C,0). Take a branch $(C_i,0)$, and delete all the stars from the resolution graph which belong to the branches $(C_j,0)$ for $j \neq i$. Call a point of the remaining graph contractible, if it satisfies one of the following conditions:

- It is not connected to the star, and connected only to points of lower weight.
- It is connected to a star, and to just one other point of lower weight.

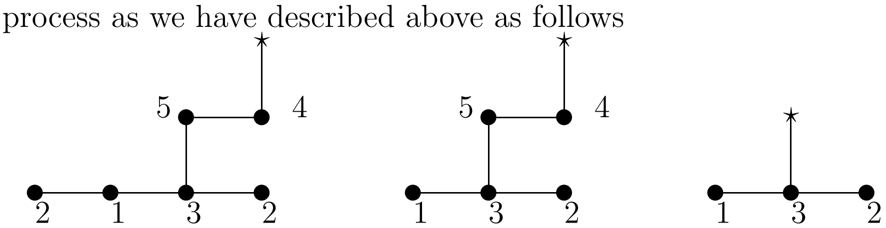
This exactly means that the corresponding curve was created in a blowing-up, which would not have been necessary for getting the standard resolution of the branch $(C_i,0)$. From the resolution process, it follows that such a point is connected to at most two points in the graph. If it is connected to just one other point, then delete the point of highest weight, and the edge connected to it. If it is connected to two points A and B say, then delete the point of highest weight, the connecting edges, and connect A and B by an edge. This corresponds to "blowing-down" the curve corresponding to this highest weight. Now iterate this process until it is no longer possible. We then arrive at the resolution graph of $(C_i,0)$. For example, if we have the following resolution graph



We can get the resolution graph of the curve corresponding to the star on he right connecting to the point with weight 4 by the following process, first delete the another two stars



and process as we have described above as follows



To get the contact number $\gamma(C_i, C_j)$, we remove all the stars corresponding to the branches C_k with $k \neq i, j$, and then successively remove points with property (1) and (2). If this is no longer possible, then we are left with the resolution graph of the union of $(C_i, 0)$ and $(C_j, 0)$. Consider the following number.

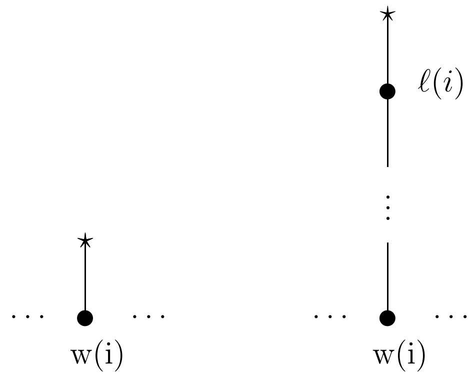
(1) The maximal natural number k such that the numbers $1, \dots, k-1, k$ occur as weight just once in the resolution graph of the union $(C_i, 0)$ and $(C_j, 0)$.

(2) The weights of the vertices connected to the stars in the resolution graph of the union of $(C_i, 0)$ and $(C_j, 0)$.

The contact number of $(C_i, 0)$ and $(C_j, 0)$ is the minimum of these numbers. So for example in the resolution graph above, the contact numbers between the curves belonging to the right two stars is equal to $min\{5, 5, 4\} = 4$, between the curve belonging to the star on the left and the other two is one.

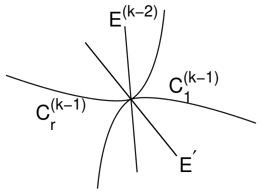
(2) $\Rightarrow$ (1) Conversely, suppose we know the resolution graph of the $(C_i, 0)$ and the contact numbers. We give an algorithm for finding the resolution graph of (C,0). The proof goes by induction on r (the number of

branches). The case r=1 is trivial. So suppose that the resolution graph $A(f)$ of $\cup_{i=1}^{r-1}(C_i,0)$ has been constructed out off the resolution graphs of the $(C_i,0)$ and the contact numbers. For $i=1,\dots,r-1$ define $\ell(i)=\gamma(C_r,C_i)$, and let l(r) be the maximum of $\gamma(C_i,C_r)$ for $i\neq r$. Let w(i) be the weight of the vertex in $A(f)$ belonging to the exceptional curve which intersects the strict transform of $(C_i,0)$, for $i=1,\dots,r-1$. Let w(r) be the vertex of highest weight in the resolution graph $A(f)_r$ of $(C_r,0)$. It is clear the point in the resolution graph connected to the star, which corresponds to $(C_i,0)$ has weight $\max\{\ell(i),w(i)\}$. So if $\ell(i)>w(i)$, we need, starting from the standard resolution, $\ell(i)-w(i)$ extra blowing-ups. Therefore, if $\ell(i)>w(i)$, we modify the graph $A(f)$ as suggested by the following picture.



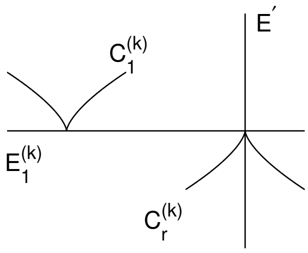
$A(f)$ $\qquad\qquad$ $A(f)'$

If $\ell(i)\leq w(i)$ we leave $A(f)$ unchanged at the star corresponding to $(C_i,0)$. We get a modified graph $A(f)'$. Similarly, we get a modified graph $A(f)'_r$. Let $k=max\{\ell(j):j<r\}$ be the maximum of the contact numbers of $(C_r,0)$ with $(C_j,0)$ for $j<r$. Without loss of generality, we may suppose that this maximum is achieved for $j=1$. Let $\pi_1\circ\ldots\circ\pi_k:X_k\longrightarrow U$ be the map which occurs in the standard resolution process of $\cup_{i=1}^{r}(C_i,0)$ after k steps. So by definition of contact number, the strict transforms $C_1^{(k)}$ and $C_r^{(k)}$ intersect a common component $E_1^{(k)}$ of $E^{(k)}$, but do not intersect each other. In fact, by choice of k, $C_r^{(k)}$ does not intersect any of the $C_j^{(k)}$. Let p be the point of $A(f)'$ which corresponds to $E_1^{(k)}$. If $k>1$, it follows from the resolution process that there is a unique component E' of $E^{(k)}$ which intersect E_1^k and was created under π_{k-1}. The point of $A(f)'$ which corresponds to E' we call q. It has weight k-1. First we show the situation in the blowing-up previous to the step where the two branches become separated:

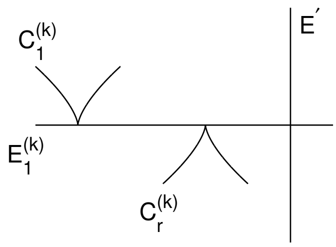


It is clear that the only two possible cases are.

(1) $k > 1$, and the curve $C_r^{(k)}$ intersect E'. Note that the points p and q in $A(f)'$ are connected by an edge. The following is a possible picture of a part of the resolution graph after k blowing-ups.



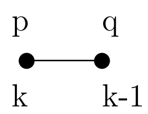
(2) k=1, or the curve $C_r^{(k)}$ does not intersect E'.



In the second case, consider the subgraph $A(f)''_r$ of $A(f)'_r$ which one gets deleting all points of weight less than k. So this is the resolution graph of $C_r^{(k-1)}$ with al weights shifted by k-1. As we already noted that none of the $C_j^{(k)}$ intersects $C_r^{(k)}$, it follows that the resolution graph of (C,0) is obtained by identifying the unique point of $A(f)''_r$ of weight k with p, which also has weight k.

In the first case, consider the subgraph $A(f)''_r$ of $A(f)'_r$ which one gets deleting all points of weight less than k-1. The graph $A(f)''_r$ looks like

and we get the resolution graph (C,0) by doing surgery: In $A(f)'_r$ one replaces the subgraph

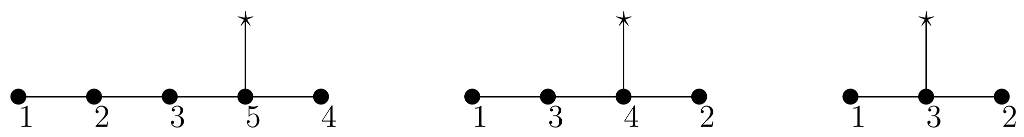


by $A(f)''_r$. So we see that we can construct the resolution graph (C,0) from the resolution graphs of $(C_i, 0)$ and the contact numbers. $\square$

Example 3.4. we consider the reducible plane curve singularity given by $f = 0$, where $f = f_1 f_2 f_3$ with

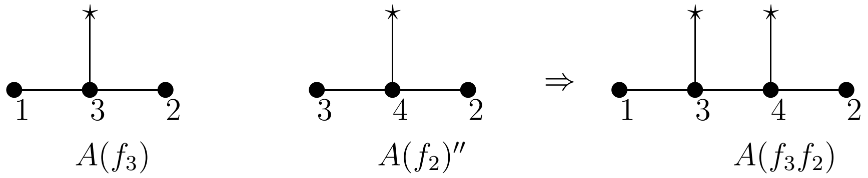
$$f_1 = x^2 - y^7,$$
$$f_2 = x^3 - y^5,$$
$$f_3 = x^2 - y^3.$$

Which have the next three resolution graphs

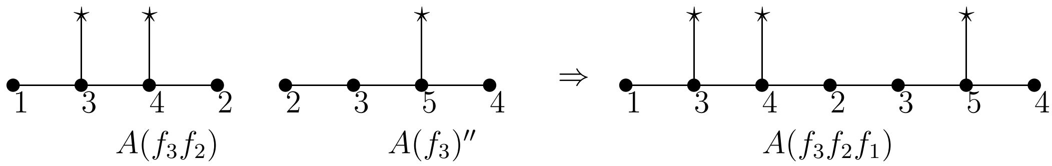


So we have to put the three resolution graphs into one. Suppose we know the contact numbers $\gamma(f_3, f_2) = 3$, $\gamma(f_3, f_1) = 2$ and $\gamma(f_1, f_2) = 2$ getting with Singular or just by hand (it is not very hard).

First suppose we have $A(f_3)$ and we want to join it to $A(f_2)$, the contact number $\gamma(f_3, f_2) = 3$, then because of the point with weight 3 in $A(f_2)$ does not have an edge joining it with the point of weight 2 in $A(f_2)$ we are in the first case of the Theorem 3.1.8 ((2) $\Leftrightarrow$ (1)), so we have:



Now we join $A(f_1)$ to $A(f_3 f_2)$, since the contact value $\gamma(f_3, f_1) = 2 = \gamma(f_1, f_2)$, k=2. Since in the graph $A(f_1)$ the points 1 and 2 are joining by an edge we are in the second case of the Theorem 3.1.8.



Finally we have a theorem that relates the multiplicity sequence and the resolution graph.

Theorem 3.1.9. *Let (C,0) be a germ of a plane curve singularity. The following data determine each other:*

(1) *The multiplicity sequence of the branches of* $(C, 0)$ *and the contact numbers between the branches.*

(2) *The resolution graphs of the branches of* $(C, 0)$ *and the contact numbers between the branches.*

(3) *The resolution graph of* $(C, 0)$*.*

proof: (1) $\Leftrightarrow$ (2) we know from the irreducible case and (2) $\Leftrightarrow$ (3) from the previous theorem. $\square$

Chapter 4

Alexander polynomial

In this chapter we are going to introduce the Alexander polynomial, and we are going to prove in the irreducible case a formula by Eisenbud-Neumann which expresses the Alexander polynomial in terms of the resolution graph. Also we are going to notice that another polynomial, the so called zeta-function of monodromy transformation of a plane curve singularity (C,0), may be derived from the Alexander polynomial, just by substituting in the Alexander polynomial all the variables by another one.

4.1 Definition

Let $(C,0) \subset (\mathbb{C}^2,0)$ be a reduced plane curve singularity with equation {f=0}, and let $C = \cup_{i=1}^{r} C_i$ be its representation as the union of irreducible components, i.e. $(C_i,0)$ is an irreducible plane curve singularity. Consider $\mathbb{C}\{x,y\}$ the ring of convergent power series and $\mathcal{O}_C = \mathbb{C}\{x,y\}/(f)$ the local ring of the curve (C,0).

We denote by $\Gamma = \Gamma(C,0) \subseteq \mathbb{Z}^r_{\geq 0}$, the semigroup associated to the plane curve singularity (C,0), introduced in the Definition (3.1.1). There exists another semigroup in $\mathbb{Z}^r_{\geq 0} \times (\mathbb{C}^*)^r$ associated to the plane curve singularity necessary in order to define the Alexander polynomial.

Definition 4.1.1. *With the notation from above we consider $(x_i(t_i), y_i(t_i))$ a primitive parametrization (for example Puiseux parametrization) of the i-th branch, and g an arbitrary element of $\mathbb{C}\{x,y\}$.*

For

$$g(x_i(t_i), y_i(t_i)) = a_i \cdot t_i^{v_i} + (terms\ of\ higher\ order).$$

we set $v_i(g) := v_i$ and $a_i(g) := a_i$, and if $g(x_i(t), y_i(t)) = 0$ we set $v_i(g) := \infty$ and $a_i(g) := 0$.

*We define, $\hat{\Gamma} = \hat{\Gamma}(C, 0)$, **the extended semigroup** of the plane curve singularity (C,0) as, the semigroup of $\mathbb{Z}^r_{\geq 0} \times (\mathbb{C}^*)^r$ which consists of elements of the form, $(\underline{v}(g); \underline{a}(g)) = (v_1(g), \cdots, v_r(g); a_1(g), \cdots, a_r(g))$ for $g \in \mathbb{C}\{x, y\}$, with $v_i(g) < \infty$, for $i = 1, \cdots, r$.*

Related with the extended semigroup we define some important concepts.

Definition 4.1.2. *Let (C,0) be a plane curve singularity, and $\hat{\Gamma}(C, 0)$ its extended semigroup, we define:*

- ***(1)*** *$\pi : \hat{\Gamma}(C, 0) \longrightarrow \mathbb{Z}^r$ the natural projection: $(\underline{v}, \underline{a}) \longrightarrow \underline{v}$.*
- ***(2)*** *For an element $\underline{v} \in \mathbb{Z}^r$, $F_{\underline{v}} = \pi^{-1}(\underline{v}) \subset \mathbb{Z}^r_{\geq 0} \times (\mathbb{C}^*)^r$, the corresponding fibre over $\underline{v}$ in the extended semigroup.*

In particular we have an easy property for the second definition.

Proposition 4.1.3. *Let (C,0) be a plane curve singularity, and let $\Gamma(C, 0)$ be its semigroup. Consider $\underline{v} \in \mathbb{Z}^r$, and $F_{\underline{v}}$ as in the last definition. Then, the fibre $F_{\underline{v}}$ is not empty iff $\underline{v} \in \Gamma(C, 0)$.*

Proof:It is trivial by construction. □
Still we need to introduce some more concepts for our purpose.

Definition 4.1.4. *Let (C,0) be a plane curve singularity defined by $f \in \mathbb{C}\{x, y\}$, and $\mathcal{O}_C$ its local ring. Consider $\underline{v} \in \mathbb{Z}^r$, we define:*

$$J(\underline{v}) = \{g \in \mathcal{O}_C \mid v_i(g) \geq v_i; i = 1, \ldots, r\},$$

that is an ideal in $\mathcal{O}_C$, and is well defined since $v_i(g_1) = v_i(g_1 + h \cdot f)$ for all $g_1, h \in \mathbb{C}\{x, y\}$. Associated to $J(\underline{v})$ we define a natural $\mathbb{C}$-linear map:

$$j_{\underline{v}} : J(\underline{v}) \longrightarrow \mathbb{C}^r; \quad g \longrightarrow (a_1, \cdots, a_r),$$

where a_i is the coefficient in the power series expansion $g(x_i(t_i), y_i(t_i)) = a_i t_i^{v_i} + \cdots$ (the number a_i may be 0). Finally we define $C(\underline{v}) \subset \mathbb{C}^r$, to be the image of the map $j_{\underline{v}}$, and $c(\underline{v}) = dim_{\mathbb{C}}\ C(\underline{v})$.

We are interested in $c(v)$, so we are going to see another point of view.

Remark 4.1. With the above notation let us consider $ker(j_{\underline{v}})$, by definition $ker(j_{\underline{v}}) = \{g \in J(\underline{v}) \mid j_{\underline{v}}(g) = 0\}$, since $g \in J(\underline{v})$ we have that $v_i(g) \geq v_i$, $i = 1, \ldots, r$, then $v_i(g) > v_i$ iff $j_{\underline{v}}(g) = 0$, thus $ker(j_{\underline{v}}) = J(\underline{v} + \underline{1})$, where $\underline{1} = (1, \ldots, 1)$. Then $C(\underline{v}) \cong J(\underline{v})/J(\underline{v} + \underline{1})$.

In particular for r=1, $c(v) = dim_{\mathbb{C}} J(v)/J(v+1)$, then:

$$c(v) = \begin{cases} 0 & \text{iff } v \notin \Gamma(C,0). \\ 1 & \text{iff } v \in \Gamma(C,0). \end{cases}$$

Example 4.1. Let (C,0) be a plane curve singularity defined by $f = f_1 \cdot f_2$ where $f_1 = y^2 - x^3$ and $f_2 = y^3 - x^5$. Consider $(x_1(t), x_2(t)) = (t^2, t^3)$ a parametrization of f_1 and $(y_1(t), y_2(t)) = (t^3, t^5)$ a parametrization of f_2. If we take $\underline{v} = (2,3) \in \mathbb{Z}^2$, then $J_{\underline{v}}$ is given by:

$$J_{\underline{v}} = \{g \in \mathcal{O}_C \mid v_1(g) \geq 2, v_2(g) \geq 3\},$$

where $g(x_1(t), x_2(t)) = a_{v_1} t^{v_1}$+higher order, $g(y_1(t), y_2(t)) = a_{v_2} t^{v_2}$+higher order, such that $v_1(g) \geq 2, v_2(g) \geq 3$. We conclude that $J_{\underline{v}} = \mathcal{O}_C \setminus \mathbb{C}$.

Now we consider $j_{\underline{v}} : \mathcal{O}_C \setminus \mathbb{C} \longrightarrow \mathbb{C}^2$, if we consider $g_1 = y^2 - x^3$ then $g_1(x_1(t), x_2(t)) = 0$ and $g_1(y_1(t), y_2(t)) = t^{10} - t^9$, this implies that $j_{\underline{v}}(g_1) = (0, -1)$, in the same way we obtain $j_{\underline{v}}(g_2) = (1, 0)$ for $g_2 = y^3 - x^5$. From here we deduce that $C(\underline{v}) = \mathbb{C}^2$, thus the dimension $c(\underline{v}) = 2$.

Let us define the concept that motivates this chapter.

Definition 4.1.5. *Let (C,0) be a plane curve singularity, and $c(\underline{v}) = dim_{\mathbb{C}} C(\underline{v})$ as in the previous definition. First we define a Laurent series given by:*

$$L_C(t_1, \cdots, t_r) = \sum_{\underline{v} \in \mathbb{Z}^r} c(\underline{v}) \cdot t^{\underline{v}}.$$

This Laurent series is infinitely long in all directions, since $c_{\underline{v}}$ can be positive for $\underline{v}$ with negative components v_i as well.

Then we define:

$$P'_C(t_1, \cdots, t_r) = (1 - t_1) \cdot \ldots \cdot (1 - t_r) \cdot L_C(t_1, \cdots, t_r).$$

Finally we define the ***Alexander polynomial*** *as follows:*

$$P_C(t_1, \cdots, t_r) = \begin{cases} P'_C(t_1, \cdots, t_r)/(1 - t_1 \cdot \ldots \cdot t_r) & for\ r > 1. \\ P'_C(t) & for\ r = 1. \end{cases}$$

We have to see whether P_C is a polynomial, to do this we show first in the remark that P'_C is a polynomial.

Remark 4.2. Let P_C be the Alexander polynomial of the plane curve singularity (C,0), and P'_C defined as before. If we take v'_i, v''_i, both positive or both negative and large enough, then $c(v_1, \cdots, v'_i, \cdots, v_r) = c(v_1, \cdots, v''_i, \cdots, v_r)$. This implies that P'_C is a polynomial.

The next proposition shows us when P_C is really a polynomial.

Proposition 4.1.6. *For a plane curve singularity (C,0), and $r > 1$, the polynomial P'_C is divisible by $(1 - t_1 \cdot \ldots \cdot t_r)$, i.e. the power series P_C is, in fact, a polynomial.*

Proof: See reference [5]. □

4.1.1 Eisenbud-Neumann formula

D.Eisenbud and W.Neumann (see reference [10]) gave a formula for the Alexander polynomial P_C of a plane curve singularity (C,0) in terms of the standard resolution of the curve. Let us see such a formula.

Theorem 4.1.7. *Let $C = \bigcup_{i=1}^{r} C_i \subset U$ be a small representative of a reducible plane curve singularity (C,0), with equation $f = \prod_{i=1}^{r} f_i \in \mathbb{C}\{x, y\}$, and branches $(C_1, 0), \ldots, (C_r, 0)$, each branch $(C_i, 0)$ has equation f_i. Let*

$$\pi : X_k \xrightarrow{\pi_k} X_{k-1} \xrightarrow{\pi_{k-1}} \ldots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} U$$

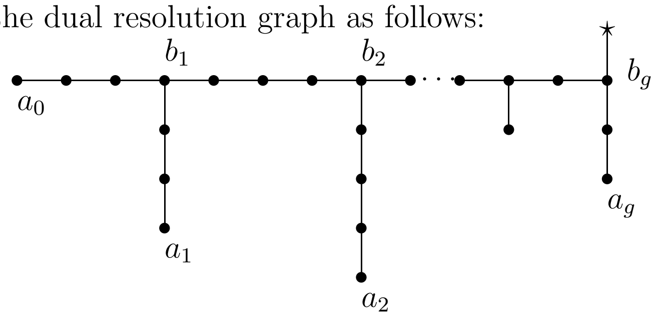
be the standard resolution of the plane curve singularity (C,0). Associated with π we consider the resolution graph, A(f), as we did in the Definition (3.1.7). Preserving the same notation we consider the exceptional divisor E, and for any vertex σ of the resolution graph consider, E_σ the component of the exceptional divisor corresponding to σ. Let $\overset{o}{E}_\sigma$ be the "smooth part" of the component E_σ, i.e. E_σ minus the intersection points with other components of the total transform of the curve (C,0). Let m_j^σ $(j = 1, 2, \cdots, r)$ be the multiplicity of the lifting $f_j \circ \pi$ of the function f_j to the space X_k of the resolution along the component E_σ, then $\underline{m}^\sigma = (m_1^\sigma, \cdots, m_r^\sigma)$. Let $\chi(\overset{o}{E}_\sigma)$ be the Euler characteristic of $\overset{o}{E}_\sigma$, that is, the number of components with whom E_σ intersects minus 2. Then we have the next equality:

$$P_C(t_1, \cdots, t_r) = \Pi_\sigma (1 - \underline{t}^{\underline{m}^\sigma})^{-\chi(\overset{o}{E}_\sigma)}, for\ r > 1. \tag{4.1}$$

$$P_C(t)/(1-t) = L_C(t) = \Pi_\sigma(1-t^{m^\sigma})^{-\chi(\overset{o}{E}_\sigma)}, for\ r=1. \tag{4.2}$$

Proof: We give a proof only in the case r=1 since this is rather elementary. The general case is a very deep result for which we refer to [10].

In this case we can reinterpret the formula in terms of the generators of the semigroup, $\beta_0, \cdots, \beta_g$. According with (4.2), $L_C(t) = \Pi_\sigma(1-t^{m^\sigma})^{-\chi(\overset{o}{E}_\sigma)}$, if we consider the dual resolution graph as follows:



The Euler characteristic $\chi(\overset{o}{E}_\sigma)$ is equal to zero unless σ is one of the vertices a_j, b_j, where $\chi(\overset{o}{E}_{a_j}) = 1$, $\chi(\overset{o}{E}_{b_j}) = -1$. Then we have to prove:

$$L_C(t) = \frac{\prod_{i=1}^{g}(1-t^{m_{a_i}})}{\prod_{i=0}^{g}(1-t^{m_{b_i}})} \quad . \tag{4.3}$$

On the other hand we know by definition,

$$L_C(t) := \sum_{i\geq 0} c_i t^i.$$

Where we have defined $c_i = dim_{\mathbb{C}}(J(i)/J(i+1))$, see Definition (4.1.4), and Remark (4.1).

Then, $L_C(t) = \sum_{\gamma\in\Gamma(C,0)} t^\gamma$. In order to prove (4.3) we have to mention the next important properties of the generators of the semigroup $\Gamma(C,0) = \langle\beta_0,\ldots,\beta_g\rangle$. Let $e_i = gcd(\beta_0,\ldots,\beta_i)$, $e_0 > e_1 > \ldots > e_{g-1} > e_g = 1$ and $N_i = \frac{e_{i-1}}{e_i}$, $i = 1,\ldots,g$.

(P1) $N_i\beta_i \subset \beta_{i+1}$.

(P2) $N_i\beta_i \in \langle\beta_0,\ldots,\beta_{i-1}\rangle$ and $\forall\gamma\in\Gamma$,

$$\gamma = \lambda_0\beta_0 + \sum_{i=1}^{g}\lambda_i\beta_i\ ;\ s.th.\ \ \lambda_i < N_i\ \ i=1,\ldots,g\ and\ \lambda_0 \geq 0$$

this writing is unique.

(P3) $m_{b_j} = \beta_j,\ j = 0, \cdots, g.$
m_{a_j} is a multiple of β_j, i.e. $m_{a_j} = N_j\beta_j,\ j = 1, \cdots, g.$

Therefore one has

$$\sum_{\gamma\in\Gamma} t^{\gamma} = \sum_{\lambda_0\geq 0\ ;\ 0\leq\lambda_i<N_i \quad i=1,\ldots,g} t^{\lambda_0\beta_0+\ \sum_{i=1}^{g}\lambda_i\beta_i} = \sum_{(\lambda_0,\ldots,\lambda_{g-1})} t^{\lambda_0\beta_0+\ldots+\lambda_{g-1}\beta_{g-1}} \sum_{0\leq\lambda_g<N_g} t^{\lambda_g\beta_g}$$

$$= \frac{t^{N_g\beta_g}-1}{t^{\beta_g}-1}\cdots\frac{t^{N_1\beta_1}-1}{t^{\beta_1}-1} \sum_{t_0\geq 0} t^{\lambda_0\beta_0} = \frac{\prod_{i=1}^{g}(1-t^{N_i\beta_i})}{\prod_{i=0}^{g}(1-t^{\beta_i})}\ .\ \square$$

Example 4.2. Let (C,0) be an irreducible plane curve singularity given by the equation $f(x,y) = y^4 - 2x^3y^2 - 4x^5y + x^6 - x^7$, as we saw in the example1.7 we know that $s(x) = x^{6/4} + x^{7/4}$ is a Y-roots of f then,

$$x(t) = t^4 \quad , \quad y(t) = t^6 + t^7 \quad , is\ \ a\ \ parameterization.$$

Hence we have the next invariants:

- **Characteristic exponents**
  $k_0 = 4 \quad , \quad k_1 = 6 \quad , \quad k_2 = 7.$
- **Generators of the semigroup**
  $\beta_0 = 4 \quad , \quad \beta_1 = 6 \quad , \quad \beta_2 = k_2 + \left(\frac{gcd(k_0)}{gcd(k_0,k_1)} - \frac{gcd(k_0,k_1)}{gcd(k_0,k_1)}\right)k_1 = 13$
- **Puiseux pairs**
  $e_0 = 4 \quad , \quad e_1 \quad , \quad e_2 = 1$ and $N_1 = \frac{e_0}{e_1} = 2 \quad N_2 = \frac{e_1}{e_2} = 2$
  then $m_1 = \frac{k_1N_1}{n} = 12/4 = 3$, $m_2 = k_2 = 7$, we have the pairs:

  $$\{(3,2),(7,2)\}$$

- **The multiplicity sequence**
  First we apply the Euclidian algorithm for $\mathcal{X}_1 = k_1 = 6$ and $r_{1,1} = k_0 = 4$

  $$6 = 1\cdot 4 + 2$$

$$4 = 2 \cdot 2$$

and now we apply it again for $\mathcal{X}_2 = k_2 - k_1 = 1$ and $r_{2,1} = r_{1,2} = 2$

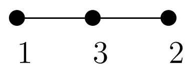
$$1 = 0 \cdot 2 + 1$$

$$2 = 2 \cdot 1$$

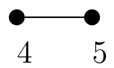
the multiplicity sequence contains: (4,2,2,1,1).

- **The resolution graph**
  The graph have $2 = g$ Puiseux chains P_1 , P_2, each one $w(1) = 2$, $w(2) = 2$ blocks (respectively). Where each chain is of the form

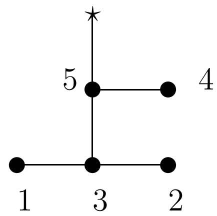
  $P_1 : \;\; S_1^1 = \{1\} \quad S_1^2 = \{2, 3\}$



  $P_2 : \;\; S_2^1 = \emptyset \qquad S_2^2 = \{4, 5\}$



  the resolution graph is



- **The Alexander polynomial**

$$P_C(t) = \frac{(1 - t^{N_1\beta_1})(1 - t^{N_2\beta_2})}{(1 - t^{\beta_0})(1 - t^{\beta_1})(1 - t^{\beta_2})} \cdot (1 - t) = \frac{(1 - t^{12})(1 - t^{26})}{(1 - t^4)(1 - t^6)(1 - t^{13})} \cdot (1 - t) =$$

$$1 - t + t^4 - t^5 + t^6 - t^7 + t^8 - t^9 + t^{10} - t^{11} + t^{12} - t^{15} + t^{16}.$$

4.1.2 Other equivalent definitions

In this section we are going to see another polynomial which coincides with the Alexander polynomial. This fact is very interesting because each one come from completely different motivations. But they are not in the center of our interest, and we are not going to describe them in details.

The first one is the classical definition of the Alexander polynomial, even though we have already used this name in our definition, the origin of this name comes from this classical definition. Since both polynomials coincide, we have adopted this name because it is the most common for everybody.

Let us see a previous definition necessary for our purpose.

Definition 4.1.8. *By a* ***link****, $L = (\Sigma, S_1 \cup \cdots \cup S_n)$, we shall mean a pair consisting of an oriented homology 3-sphere Σ, and a collection of smooth disjoint oriented simple closed curves S_i in Σ. If the ambient homology sphere Σ is the standard sphere S^3, we speak about link in S^3.*

We recall the classical definition of the Alexander polynomial, $\Delta^L(t_1, \cdots, t_n)$ of a link $L = (\Sigma, S_1 \cup \cdots \cup S_n)$.

Definition 4.1.9. *Let $\pi : \hat{\Sigma}_0 \longrightarrow \Sigma_0$ be a universal abelian cover of the link exterior, and let $\hat{p} = \pi^{-1}(p)$ be the typical fiber. The group of covering transformation is $H_1(\Sigma_0) = \mathbb{Z}^n$. Writing t_i for the covering transformation associated to an oriented meridian of S_i, the group $H_1(\hat{\Sigma}_0, \hat{p})$ becomes a module over $\mathbb{Z}[t_1, t_1^{-1}, \cdots, t_n, t_n^{-1}]$. The i-th fitting ideal $F_i = F_i(H_1(\hat{\Sigma}_0, \hat{p}))$ is the ideal in $\mathbb{Z}[t_1, t_1^{-1}, \cdots, t_n, t_n^{-1}]$ generated by $(g-i) \times (g-i)$ minors of a presentation matrix for $H_1(\Sigma_0) = \mathbb{Z}^n$, where g is the number of generators. $\Delta^L(t_1, \cdots, t_n)$ is the greatest common divisor of F_1.*

Note: For more details , see reference [10].

Now we want to explain what we mean by the Alexander polynomial of a given curve.

Definition 4.1.10. *Let (C,0) be a reduced plane curve singularity, we define $\Delta^C(t_1, \cdots, t_r)$,* ***the Alexander polynomial*** *of the link $C \cap S^3_\varepsilon \subset S^3_\varepsilon$ for $\varepsilon > 0$ small enough, where S^3_ε is the standard sphere of radius ε.*

Remark 4.3. Campillo A., Delgado F. and Gusein-Zade S.M. have proved (see reference [5]) that both definitions coincide, i.e. $\Delta^C(t_1, \cdots, t_r) = P_C(t_1, \cdots, t_r)$ for $r \geq 1$. This implies that, for a plane curve singularity (C,0), the polynomial $P_C(t_1, \cdots, t_r)$ determines the semigroup $\Gamma(C, 0)$.

We also are interested in an other polynomial, the so-called zeta-function of monodromy of the singularity (C,0).

Definition 4.1.11. *Let (C,0) be a reduced plane curve singularity defined by the equation $f \in \mathbb{C}\{x, y\}$. Let $h_f : V_f \longrightarrow V_f$ be the monodromy transformation of the singularity f acting on its Milnor fibre V_f (see reference [2]). We define* ***the zeta-function*** *of the monodromy transformation $h_f : V_f \longrightarrow V_f$ of the singularity f:*

$$\zeta_f(t) = \Pi_{q\geq 0}[det(id - t \cdot h_{f_*} \mid_{H(V_f;\mathbb{R})})]^{(-1)^{q+1}}.$$

Remark 4.4. This polynomial may be derived from the Alexander polynomial, just substituting the variables t_i by t, i.e $\zeta_f(t) = \Delta^C(t, \cdots, t)$ for $r > 1$, and $\zeta_f(t) = \Delta^C(t)/(1-t)$ for r=1.

Chapter 5

Implementation

All the theoretical development we have already done, has an important application along this chapter. We have performed in Singular some procedures to implement, the resolution graph and the Alexander polynomial. Now, what we are interested in is to explain the most important procedures and to give some examples to show how to work with them.

5.1 Important procedures

Let us study the procedures related with the resolution graph. In our procedures we do not return a graph but a matrix, the so called incidence matrix, representing the resolution graph in the following way.

The **incidence matrix** of a reducible reduced plane curve singularity $(C,0) = \bigcup_{i=1}^{r}(C_i,0)$, is a symmetric matrix of integer numbers associated to the resolution graph of (C,0). Suppose we have a graph with n vertices,say $\sigma_1, \cdots, \sigma_n$, then the incidence matrix is a $n \times n$ matrix $(a_{i,j})$ with

$$a_{i,j} = \begin{cases} 1 & if\ i \neq j\ and\ \sigma_i\ is\ joined\ with\ \sigma_j\ by\ an\ edge. \\ 0 & if\ i \neq j\ and\ \sigma_i\ is\ not\ joined\ with\ \sigma_j\ by\ an\ edge. \\ -k & if\ i = j\ and\ \sigma_i\ corresponds\ to\ the\ total\ transform\ of\ (C_k,0). \\ k & if\ i = j\ and\ \sigma_i\ corresponds\ to\ the\ exceptional\ divisor\ with\ weight\ k. \end{cases}$$

The first procedure calculates the incidence matrix of the resolution graph of each branch, and also the total multiplicities in the corresponding standard resolution of each graph.

Algorithm 5.1.1. [decomp-irred-graphs]
INPUT: *F, a bivariate polynomial defining a plane curve singularity.*
OUTPUT: result a list containing, the resolution graph of each branch, together with the total multiplicities of each branch.

Start

(1) *With F get the characteristic exponents of each branch.*

(2) *FOR each branch, with its characteristic exponents DO perform the algorithm described in Theorem 2.2.8 (in alexpoly.lib called Mdividends) and save in the matrix N, in the i-th row, the quotients $\mu_{i,j}$ of the i-th step.*

(3) *Create a matrix M= $M_{n\times n}$, where $n = \sum_{i,j} \mu_{i,j}$.*

(4) *FOR each matrix N in the step (2), i.e for each branch DO (with the same notation as in Theorem 2.2.9)*

- *FOR $i = 1, \ldots,$number of rows in N DO create a Puiseux chains P_i as follows:*
  - *FOR $j = 1, \ldots,$ number of columns in N DO create a building block S_i^j as follows:*
    - *S_i^j has such a points as indicate the position $n_{i,j}$ of the matrix N, and numerate the points starting from $\pi(i,j) + 1$ until $\pi(i,j) + \mu_{i,j}$.*
    - *Put S_i^j in M, i.e. the position $m_{\pi(i,j)+k,\pi(i,j)+k+1} = 1$ and also the symmetric, for $k = 1, \ldots, \mu_{i,j}$.*
    - *IF $j > 2$ THEN Connect in M, the end point of S_i^{j-2} with the start point of S_i^j, i.e. in the corresponding positions of M write 1.*
    - *IF j=last column in N different to 0 DO Connect in M, the end point of S_i^{j-1} with the end point in S_i^j, i.e. in the corresponding positions of M write 1.*
    - *Connect different chains. IF $i > 1$ DO*

i) *IF j=1 and* $m_{j,k} <> 0$ *DO*
Connect the end point of $S_i^{w(i)}$ *with the start point of* S_{i+1}^1.

ii) *IF j=3 and* $m_{j,1} = 0$ *DO*
Connect the end point of $S_i^{w(i)}$ *with the start point of* S_{i+1}^3.

iii) *IF j=2 and* $m_{j,1} = 0$ *DO*
Connect the end point of $S_i^{w(i)}$ *with the end point of* S_{i+1}^2.

* *Give weight to the points,* $m_{i,i} = i$*, for* $i = 1, \dots, n-1$.
Give weight to the star, $m_{n,n} = -1$.

- *Calculate in the vector total-multiplicities the total multiplicities in each point. In the library alexpoly.lib this is done for irreducible curves with the procedure, totalmult.*
- *result[i]=M,total-multiplicities.*

(5) *RETURN(result).*

End

Now we use this procedure to glue all the resolution graphs and get the incidence matrix of the total curve (C,0).

Algorithm 5.1.2. [totalmultiplicities]
In this procedure we follow the Theorem 3.1.8, and the notation will be the same.
INPUT: *F is a bivariate polynomial defining a plane curve singularity.*
OUTPUT: *result a list containing, the resolution graph of the curve defined by F, together with the total multiplicities.*
Start

(1) *We call the last procedure to get the resolution graph and the total multiplicities of each branch. We save this information in the list L.*
L=decomp-irred-graphs(F).

(2) IF the number of branches is 1 DO
we have nothing to glue, result=L[1][1],L[2][1].

(3) ELSE

- *Construct a matrix called information containing: in each row we save information of each branch in order to can glue the branches. The first column say us how many points contribute each branch to the final graph. The second column say us with whom (of the before) we must glue the current branch. The third say us the contact number between the current branch and the branch indicated by the before column. The last column how many point attach if* $l(i) > w(i)$ *where l(i) is the contact indicated in the third column and w(i) is the weight of the vertex in the graph belong to the exceptional divisor which intersects the strict transform of the curve.*
- *Reconstruct the matrix of each graph,* Γ' *and* Γ'_r*, knowing the information saved in information.*
  FOR $i = 1, \ldots,$*number of branches DO*
  - *The all weight of the point joined with the star in the i-th branch was w(i), then calculate the new weight w1(i), that should be the last one plus the points indicated in information[i,4].*
  - *The new matrix is a quadrat matrix of size,* $(w1[i]+1) \times (w1[i]+1)$*, w1[i] points and the star.*
  - *The new points are consecutive. In the i-th matrix, say* M_i*, we put* $M_i[w[i], w[i]+k] = 1$ *for* $k = 1, \ldots, w1[i]-w[i]$*, also for the symmetric* $M_i[w[i]+k, w[i]] = 1$*, and join the last one with the star* $M_i[w1[i], w1[i]+1] = 1$*.*
  - *Finally give weight to each point,* $M_i[k,k] = k$ *for* $k = 1, \ldots, w1[i]$*, and to the star a negative number, in each branch different to distinguish* $M_i[w1[i]+1, w1[i]+1] = -i$*.*

- *Create the final matrix $M = M_{n\times n}$ of the total graph, where $n = \sum_j information[j,1]$.*
- *Insert the first matrix M_1 in M, in the first positions of the matrix M.*
- *For $i = 2, \ldots,$number of branches DO*
  - *IF the i-th branch do not separate in the first step DO*
    We are interesting in the crucial position k=information[i,3], check whether point with weight k and the point with weight k-1 are joined by an edge.
    - *IF $M_i[k, k-1]$ is different to 1 DO*
      - *. We have to join the i-th branch with the branch j=information[i,2].*
      - *. Find the position of the point of weight k in M corresponding to the j-th branch, we call "a" to this position.*
      - *. Disconnect the position a with the a-1, $M[a, a-1] = 0$, (and the same for the symmetric).*
      - *. Find the position of the point of weight k-1 and k in M corresponding to the i-th branch, we call "b" and to "c" respectively to this positions.*
      - *. Insert in M the matrix M_i, but only the points with weight bigger than k-1.*
      - *. Connect in M the points of weight k-1 corresponding to M_j and M_i, i.e M[b,a-1]=1, and the same for the points of weight k, M[c,a]=1, (also for the symmetric).*
    - *ElSE*
      - *. We have to join the i-th branch with the branch j=information[i,2]-th.*
      - *. Insert in M the M_i, but not all the points, only the point of weight bigger than k.*
      - *. Find the position of the point of weight k in M corresponding to the j-th branch, we call "a" to this position.*

. *Find all the points of the i-th branch that was joined to the point with weight k in M_i, then find the new position in M of such a points, say "b" (b is different for each point), and join them with point in M of weight k coming from M_j, i.e. M[b,a]=1, (the same for the symmetric).*

∗ *ELSE(the graphs separated in the first step)*

- *We have to join the i-th branch with the branch j=information[i,2]-th.*
- *Check which point in M_i is joined with the point of weight 1, say "p" to such a point.*
- *Put in M all the points from M_i with weight higher than 1.*
- *Calculate the new position of p in M, say b, and the new position in M of the point with weight 1 that come from the matrix M_j, say a.*
- *Join in M the positions a and b, i.e. M[b,a]=1, (and the same for the symmetric).*

- *Calculate a vector, say totalmult, with the total multiplicities of the whole graph, since the total multiplicities of each graph, that we know them, L[2][i] for $i = 1, \ldots,$number of branches. (In the library alexpoly.lib with procedure insertmult).*
- *result=M,totalmult.*

(4) RETURN(result).

End

As we have seen the procedure totalmultiplicities return the resolution graph and the total multiplicities, but may be we are only interested in the resolution graph, for this we have created the following procedure.

Algorithm 5.1.3. [**resolutiongraph**]
INPUT: F is a bivariate polynomial defining a plane curve singularity.
OUTPUT: M, the incidence matrix of the resolution graph.
Start
return(totalmultiplicities(F)[1]).
End

Algorithm 5.1.4. [alexanderpolynomial]
In this procedure we follow the Theorem 4.1.7.
INPUT: *F is a bivariate polynomial defining a plane curve singularity.*
OUTPUT: A list containing a ring with the Alexander polynomial and the zeta function of the monodromy.
Start

(1) *L=totalmultiplicities(F). In L[1] we have the resolution graph, and in L[2] the total multiplicities.*

(2) *Generate a vector V, where we compute the Euler characteristic in each point of the resolution graph. For each point we consider the row in the incidence matrix that represent such a point the the Euler characteristic is the number of elements different to 0 in the corresponding row except the diagonal, minus 2.*

(3) *Create a ring ALEXring, with characteristic the characteristic of the basering, variables(t,t(1..g)) where g is the number of columns in the matrix of total multiplicities L[2], i.e. the number of branches of the curve defined by F, and any order* $>$.

(4) *Compute g following the formula in the Equation (4.1),* $\Pi_\sigma(1-\underline{t}^{\underline{m}^\sigma})^{-\chi(\overset{o}{E}_\sigma)}$.

- *FOR* $i = 1, \ldots,$ *number of rows in L[2] DO*
  - *h=1.*
  - *FOR* $j = 1, \ldots,$ *number of columns in L[2] DO*

    $$h = h \cdot t(b)^{L[2][i,j]}.$$

  - *IF h is different to 1 DO*
    If h=1, the corresponding point is really a star and do not apport anything.
    - *IF* $V[i] < 0$ *DO*
      . $h = (1-h)^{((-1)\cdot V[i])}$.
      . $denominator = denominator \cdot h$.

- *ELSE*
  . $h = (1-h)^{V[i]}$.
  . $numerator = numerator \cdot h$.

- *IF number of branches is 1 DO*
  * $numerator = numerator \cdot (1 - t(1))$.
- *g=numerator/denominator.*

(5) *Perform the zeta function of the monodromy transformation of a plane curve singularity (C,0).*

- *zeta-monodromy=g(t,t,...,t).*

(6) *EXPORT(g).*
EXPORT(zeta-monodromy).

End

5.2 Other procedures

We have also performed three more procedures. The first calculates the multiplicity sequence from the characteristic exponents, this procedure is called charexp2multseq and it is used in the procedure decomp-irred-graphs.

The other two are not used in any procedure, but we consider them very interesting because they are completely new when we transform, one invariant into another, inside Singular. One of them calculates the characteristic exponents from the multiplicity sequence, and is called multseq2charexp. The last one calculate an irreducible polynomial with given characteristic exponents. This procedure is very useful to generate examples, and it is done following the description in the reference [10].

Algorithm 5.2.1. [charexp2multseq]
INPUT: *V, the characteristic exponents of an irreducible plane curve singularity (C,0).*
OUTPUT: *W, the multiplicity sequence of (C,0).*
Start

(1) *IF number of characteristic exponents is 1 DO*

- *W=1.*
- *RETURN(W).*

(2) - *Use the Theorem 2.2.8 applied to the characteristic exponents V[2] and V[1], then we get $\mu_{1,i}$ and $r_{1,i}$, for $i = 1, \ldots, w(1)$.*
  - *FOR $i = 1, \ldots, w(1)$ DO*
    - *W=W,($\mu_{1,i}$ times $r_{1,i}$).*

(3) *FOR $j = 3, \ldots,$ number of characteristic exponents DO*

- *The same that in the step (2), applied each time to V[i] and V[i-1] , at the end we have in W the multiplicity sequence of (C,0).*

(4) *RETURN(W).*

End

Algorithm 5.2.2. [multseq2charexp]
INPUT: *W, the multiplicity sequence of an irreducible plane curve singularity (C,0).*
OUTPUT: *V, the characteristic exponents of (C,0).*
Start

(1) *IF size of W is different to 1 DO*

- *Count how many consecutive times is repeated W[1] in the multiplicity sequence W, call r_1 to this number.*
- *Calculate the two first characteristic exponents.*
  - *V[1]=W[1].*
  - *V[2]=$r_1 \cdot W[1] + W[1 + r_1]$.*
- *Perform the Euclides algorithm with V[2] and V[1], in order to get the position in W, say l, where should start the multiplicities corresponding to the next Euclides algorithm with V[3] and V[2], (if exist V[3]).*
- *temp=1.*
- *WHILE l<cardinal of W minus 1 DO*

* temp=temp+1.
* Count how many consecutive times is repeated W[temp] in the multiplicity sequence W, call r_{temp} to this number.
* $V[temp+1] = V[temp] + (r_{temp} \cdot V[l]) + V[r_{temp} + l + 1]$.
* Calculate the new value of l wit the Euclides algorithm as before.

(2) ELSE

- V=1.

(3) RETURN(V).

End

Algorithm 5.2.3. [charexp2poly]
INPUT: *V, the characteristic exponents of an irreducible plane curve singularity (C,0). A, the coefficients in the parametrization $x(t) = t^{V[1]}; y(t) = \sum_{j>2} A_j t^{V[j]}$.*
OUTPUT: *f, a polynomial representing the irreducible plane curve singularity (C,0), with characteristic exponents V.*
Start

(1) ring B=0,(x,y),lp.

(2) FOR $i = 2, \cdots,$ size of V.

- s=v[i] mod v[1].
- $w(s+1) = w(s+1) - A[i-1] * x^((V[i] - s)/V[1])$.

(3) Construct the matrix that generate the polynomial.

- M is a matrix of size $V[1] \times V[1]$.
- FOR $i = 1, \ldots, V[1]$ DO
  * M[i,i]=y+w[1].
  * FOR $j = 1, \ldots, V[1] - i$ DO
    - M[j,j+i]=w[i+1].
  * FOR $j = 1, \ldots, V[1] - i$ DO
    - M[j+i,j]=x·w[1+v[1]-i].

(4) f=determinant(M).

(5) RETURN(f).

End

Appendix A

User manual

A.1 The library alexpoly.lib

Prerequisites: This library makes usage of the Singular library hnoether.lib.
Purpose: alexpoly.lib contains procedures to compute:
characteristic exponents.
Given a reduced plane curve singularity (C,0).

- The incidence matrix of (C,0), together with the total multiplicities.
- The Alexander polynomial of (C,0), and the zeta function of the monodromy.

  If we suppose (C,0) is irreducible.

- Given the vector of characteristic exponents of (C,0), calculate the multiplicity sequence of (C,0).
- Given the multiplicity sequence of (C,0), calculate the characteristic exponents of (C,0).
- Given the vector v of characteristic exponents of (C,0), the procedure calculates an equation of some polynomial which has this characteristic exponents.

A.1.1 Example Session

Let (C,0) be an irreducible plane curve singularity given by a bivariate polynomial $f \in \mathbb{C}\{x, y\}$, i.e. $\mathbb{C}\{x, y\}/(f)$ is the local ring of the singularity, and f is irreducible as power series over $\mathbb{C}$.

Let us show how to compute in Singular the incidence matrix of the resolution graph. One possible command to compute such an intmat is resolutiongraph,

```
proc resolutiongraph (def INPUT)

USAGE:    resolutiongraph(INPUT);INPUT poly or list.

ASSUME:  INPUT is either a bivariate polynomial defining a plane
         curve singularity, or the output of hnexpansion(f[,\"ess\"]),
         or the list hne in the ring created by the
         hnexpansion(f[,\"ess\"]), or the output of develop(f)
         resp. of extdevelop(f,n), or a list containing the contact
         matrix and a list of integer vectors with the
         characteristical exponents of the branches of a plane
         curve singularity, or an integer vector containing the
         characteristical exponents of an irreducible plane
         curve singularity.
RETURN: intmat, the incidence matrix of the resolution graph of
         the plane curve defined by INPUT, where the entries on
         the diagonal are the weights of the vertices of the
         graph and a negative entry corresponds to the strict
         transform of a branch of the curve.
NOTE:    In case the Hamburger-Noether expansion of the curve
         f is needed for other purposes as well it is better to
         calculate this first with the aid of hnexpansion and
         use it as input instead of the polynomial itself.
```

with syntaxis as follows,

```
 LIB "alexpoly.lib";
 ring A=0,(x,y),ls;
 poly f=x3-y5;
 list l=resolutiongraph(f);
```

```
 l;
  [1]:
   1,0,1,0,0,
   0,2,0,1,0,
   1,0,3,1,0,
   0,1,1,4,1,
   0,0,0,1,-1
```

The output is a list containing only one entry where is the incidence matrix. Also we can use another procedure but with an extra information, the total multiplicities of the irreducible plane curve singularity.

```
proc totalmultiplicities (def INPUT)

USAGE:   totalmultiplicities(INPUT); INPUT poly or list.

ASSUME: INPUT is either a bivariate polynomial defining a plane
         curve singularity,or the output of hnexpansion(f[,\"ess\"]),
         or the list hne in the ring created by
         hnexpansion(f[,\"ess\"]), or the output of develop(f)
         resp. of extdevelop(f,n), or a list containing the
         contact matrix and a list of integer vectors with the
         characteristical exponents of the branches of a plane
         curve singularity, or an integer vector containing the
         characteristical exponents of an irreducible plane curve
         singularity.

RETURN:  list L of two integer matrices. L[1] is the incidence
         matrix of the resolution graph of the plane curve
         defined by INPUT, where the entries on the diagonal are
         the weights of the vertices of the graph and a negative
         entry corresponds to the strict transform of a branch of
         the curve. L[2] is an integer matrix,which has for each
         vertex in the graph a row and for each branch of the
         curve a column. The entry in position [i,j] contains the
         total multiplicity of the j-th branch, i.e. the branch
         with weight -j in L[1] along the exceptional divisor
         corresponding to the i-th row in L[1]. In particular,
```

```
         the i-th row contains the total multiplicities of the
         branches of the plane curve (defined by INPUT) along the
         exceptional divisor which corresponds to the i-th row in
         the incidence matrix L[1].

NOTE:    The total multiplicty of a branch along an exceptional
         divisor is the multiplicity with which this exceptional
         divisor occurs in the total transform of this branch
         under the resolution corresponding to the resolution graph.
         In case the Hamburger-Noether expansion of the curve f is
         needed for other purposes as well it is better to calculate
         this first with the aid of hnexpansion and use it as input
         instead of the polynomial itself.
```

```
 poly f=x16-6x11y-x9-2x8y3+3x6y2-3x3y4+y6;
 list l=totalmultiplicities(f);
 l;
[1]:
   1,0,1,0,0,0,0,0,0,
   0,2,1,0,0,0,0,0,0,
   1,1,3,1,0,0,0,0,0,
   0,0,1,4,1,0,0,0,0,
   0,0,0,1,5,0,0,1,0,
   0,0,0,0,0,6,1,0,0,
   0,0,0,0,0,1,7,1,0,
   0,0,0,0,1,0,1,8,1,
   0,0,0,0,0,0,0,1,-1
[2]:
   6,9,18,21,24,25,50,75
```

The list l consists in two entries, l[1] contain the incidence matrix and in l[2] the totalmultiplicities of the irreducible singularity.

We have used the procedures resolutiongraph and totalmultiplicities with input a polynomial, but also is allowed in both as input either an intvec with the characteristic exponents of the curve or a list hne where is the output of the procedure develop(or extdevelop). For example v=4,6,7 are the characteristic exponents of the curve $f = y4 - 2x3y2 - 4x5y + x6 - x7$.

```
 intvec v=4,6,7;
```

```
 resolutiongraph(v);
 1,0,1,0,0,0,
 0,2,1,0,0,0,
 1,1,3,0,1,0,
 0,0,0,4,1,0,
 0,0,1,1,5,1,
 0,0,0,0,1,-1
```

or using develop,

```
list hne=develop(x2-y7);
 resolutiongraph(hne);
1,1,0,0,0,0,
1,2,1,0,0,0,
0,1,3,0,1,0,
0,0,0,4,1,0,
0,0,1,1,5,1,
0,0,0,0,1,-1
```

Now suppose we now the multiplicity sequence of f, for example we can get it in Singular with the next command:

```
poly g=-x11+x10-4x8y-2x5y2+y4;
list hne=develop(g);
multsequence(hne);
4,4,2,2,1,1
```

The multiplicity sequence of g is (4,4,2,2,1,1), we have a procedure that give us the characteristic exponents since the multiplicity sequence, and is described as follows,

```
proc  multseq2charexp(list L)

USAGE:   multseq2charexp(L),list L.

ASSUME:  The input is a list where in each entry is the
         mutiplicity sequence of an irreducible plane curve singularity .
RETURN:  A list where in each entry is the characteristic
         exponents of the irreducible plane curve singularity
         corresponding to the same entry in the input.
```

let us show the syntaxis:

```
intvec v=4,4,2,2,1,1;
list L=v;
multseq2charexp(L);

[1]://the characteristic exponents
  4,10,11
```

Where the output is a list, where in the first entry is the vector of characteristic exponents.

We can also make the inverse process, i.e. since the characteristic exponents get the multiplicity sequence, this process is done with the procedure charexp2multseq(v),

```
proc charexp2multseq (intvec charexp)

USAGE:  charexp2multseq(v), v intvec.

ASSUME: v contains the characteristic exponents of an
        irreducible plane curve singularity.
RETURN: intvec, the multiplicity sequence of the plane curve
        singularity.
```

We can compute this with the next command,

```
intvec v=4,10,11;
charexp2multseq(v);
4,4,2,2,1,1
```

The output is a vector containing the multiplicity sequence.

It is possible given the characteristic exponents of an irreducible plane curve singularity to calculate an irreducible polynomial f such that f has those characteristic exponents. The procedure is charexp2poly

```
proc charexp2poly(intvec v,intvec a)

USAGE:  charexp2poly(v,a); intnvec v,a.
```

```
ASSUME:  v an intvec containing the characterictic exponents of a
         irreducible plane cureve singularity.
         a an intvec with the coefficients of a parametrization given by
         x(t)=x^v[1], y(t)=a(1)t^v[2]+...+a[n-1]t^v[n], i.e. the entries
         of a are of the type number .
RETURN:  A polynomial f in the first to variables of the basering,
         such that f define an irreducible plane curve singularity
         with characteristic exponents v.
NOTE:    The entries in a should be of the type number and the
         vector v should be the sequence of characteristic exponents
         of an irreducible plane curve singularity with charcteristic
         exponents v.
```

Given (8,12,14,17) as characteristic exponents, then may compute f:

```
intvec v=8,12,14,17;
intvec a=1,1,1;
charexp2poly(v,a);
-x17+8x16-20x15+17x14-16x13y+12x12y2-2x13+32x12y-16x11y2
-8x10y3+x12-8x11y+20x10y2-16x9y3-4x9y2+16x8y3-2x7y4-8x6y5
+6x6y4-8x5y5-4x3y6+y8.
```

And finally we have a procedure to compute the Alexander polynomial associated to f in the variable t(1) and the zeta function of the monodromy operator of f in the variable t.

```
USAGE:  alexanderpolynomial(INPUT); INPUT poly or list.

ASSUME: INPUT is either a bivariate polynomial defining a plane
        curve singularity, or the output of
        hnexpansion(f[,\"ess\"]), or the list hne in the ring
        created by hnexpansion(f[,\"ess\"]), or the output of
        develop(f) resp. of extdevelop(f,n), or a list containing
        the contact matrix and a list of integer vectors with the
        characteristical exponents of the branches of a plane curve
        singularity, or an integer vector containing the
        characteristical exponents of an irreducible plane curve
        singularity.
```

```
CREATE: a ring with variables t, t(1), ..., t(r) (where r is the
        number of branches or the plane curve singularity f defined
        by INPUT) and ordering ls over the ground field of the basering.
        Moreover, the ring contains the Alexander polynomial of f in
        variables t(1), ..., t(r) alexpoly the zeta function of the
        monodromy operator of f in the variable t zeta_monodromy,
        and a list containing the factors of the Alexander polynomial
        with multiplicities alexfactors.
RETURN: a list, say ALEX, where ALEX[1] is the created ring.

NOTE:   to use the ring type: list ALEX=alexanderpolynomial();
        def ALEXring=ALEX[i]; setring ALEXring; Alternatively
        you may use the procedure sethnering and type:
        sethnering(ALEX,\"ALEXring\");
        To access the Alexander polynomial resp the zeta function:
         alexpoly resp. zeta_monodromy resp alexfactors.
        In case the Hamburger-Noether expansion of the curve f is
        needed for other purposes as well it is better to calculate
        this first with the aid of hnexpansion and use it as input
        instead of the polynomial itself.
```

Let us see an example to show how to use it.

```
poly f=-x9+x8-6x7y+3x6y2-2x4y3-3x3y4+y6;
 list ALEX=alexanderpolynomial(f);
 def alexring=ALEX[1];
 setring(alexring); alexpoly;
t(1)^36-t(1)^35+t(1)^30-t(1)^29+t(1)^28-t(1)^27+t(1)^24-t(1)^23
+t(1)^22-t(1)^21+t(1)^20-t(1)^19+t(1)^18-t(1)^17+t(1)^16-t(1)^15
+t(1)^14-t(1)^13+t(1)^12-t(1)^9+t(1)^8-t(1)^7+t(1)^6-t(1)+1.
 alexnumerator;
-t(1)^75+t(1)^74+t(1)^51-t(1)^50+t(1)^25-t(1)^24-t(1)+1.
 alexdenominator;
-t(1)^39+t(1)^33+t(1)^31-t(1)^25+t(1)^14-t(1)^8-t(1)^6+1.
```

Notice that when we execute the procedure alexanderpolynomial this creates a ring. To get the alexpoly we have to change the ring to the ring we have created with the procedure and inside this ring we have the zeta function (put: zeta_monodromy;), the alexander polynomial (put: alexpoly;), the numerator of the alexander polynomial (put: alexnumerator;), the denominator of the alexander polynomial (put: alexdenominator;).

Now we come to the case of reducible curves. Essentially we have the same procedures as in the irreducible case, i.e. resolutiongraph, totalmultiplicities and alexanderpolynomial. Let us see some examples:

First with input a polynomial and using resolutiongraph, the output is just a list with one entry where is the incidence matrix of the whole graph.

```
 poly f=(x-y)*(x5-y7);
 resolutiongraph(f);
1,1,0,0,1,0,0,
1,-1,0,0,0,0,0,
0,0,2,1,0,0,0,
0,0,1,3,0,1,0,
1,0,0,0,4,1,0,
0,0,0,1,1,5,1,
0,0,0,0,0,1,-2
```

Second with input a list containing, the contact matrix and the characteristic exponents of each branch. Now we use the procedure totalmultiplicities, thus the output is a list with two entries, in the first is the incidence matrix of the whole graph and in the second there is an intmat where each column represents a branch of the plane curve singularity and the number of rows is the same as the number of vertices in the resolution graph, i.e. the number of rows in the incidence matrix. The entry $a_{i,j}$ is the total multiplicity of the vertex corresponding to the i-th row and the j-th column (branch).

```
 intmat m[2][2]=0,3,3,0;
 intvec v1=2,3;
 intvec v2=4,6,7;
 list vectors=v1,v2;
 list L=m,vectors;
 totalmultiplicities(L);
[1]: \\incidence matrix of the resolutiongraph
   1,0,1,0,0,0,0,
   0,2,1,0,0,0,0,
   1,1,3,1,0,1,0,
```

```
   0,0,1,-1,0,0,0,
   0,0,0,0,4,1,0,
   0,0,1,0,1,5,1,
   0,0,0,0,0,1,-2
[2]:\\ total multiplicities
   2,4,
   3,6,
   6,12,
   0,0,
   6,13,
   12,26,
   0,0
```

Third case with a list containing the output of hnexpansion, we use again the procedure totalmultiplicities, so it is clear the output.

```
list hn=hnexpansion(x*(x2-y3)*(x3-y2));
 totalmultiplicities(hn);
[1]:
   1,0,1,0,0,1,0,0,
   0,2,1,0,0,0,0,0,
   1,1,3,1,0,0,0,0,
   0,0,1,-1,0,0,0,0,
   0,0,0,0,2,1,0,1,
   1,0,0,0,1,3,1,0,
   0,0,0,0,0,1,-2,0,
   0,0,0,0,1,0,0,-3
[2]:
   2,2,1,
   3,2,1,
   6,4,2,
   0,0,0,
   2,3,2,
   4,6,3,
   0,0,0,
   0,0,0
```

The Alexander polynomial is calculated exactly equal as in the irreducible case.

```
  poly f1=(y2-x3)^2-4x5y-x7;
  poly f2=y2-x3;
  poly f3=y3-x2;
  list ALEX=alexanderpolynomial(f1*f2*f3);
  def ALEXring=ALEX[1];
  setring ALEXring;
 > alexpoly;
-t(1)^18*t(2)^36*t(3)^13-t(1)^16*t(2)^32*t(3)^10-t(1)^15*t(2)^30*t(3)^11
-t(1)^13*t(2)^26*t(3)^8+t(1)^5*t(2)^10*t(3)^5+t(1)^3*t(2)^6*t(3)^2
+t(1)^2*t(2)^4*t(3)^3+1
> zeta_monodromy;
-t^67-t^58-t^56-t^47+t^20+t^11+t^9+1
> alexnumerator;
-t(1)^23*t(2)^46*t(3)^18+t(1)^19*t(2)^38*t(3)^12
+t(1)^17*t(2)^34*t(3)^14-t(1)^13*t(2)^26*t(3)^8
+t(1)^10*t(2)^20*t(3)^10-t(1)^6*t(2)^12*t(3)^4-t(1)^4*t(2)^8*t(3)^6+1
> alexdenominator;
t(1)^5*t(2)^10*t(3)^5-t(1)^3*t(2)^6*t(3)^2-t(1)^2*t(2)^4*t(3)^3+1.
```

A.1.2 Documentation of the important procedures

```
LIBRARY: alexpoly.lib

         Principals procedures:
totalmultiplicities(f); resolution graph and totalmultiplicities
                        of f.
resolutiongraph(f);     resolution graph of f.
alexanderpolynomial(f); Alexander polynomial of f.
charexp2multseq(v);     converts the characteristic exponents to
                        the multiplicity sequence.
multseq2charexp(L);     converts the multiplicity sequence of each
                        branch of a given reducible plane curve
                        singularity into the characteristic
                        exponents of the respective branch.
charexp2poly(v,a);      calculates an irreducible polynomial f
                        such that f is the equation of an irreducible
                        plane curve singularity (C,0) with characteristic
                        exponents v.
```

Bibliography

[1] A'Campo N.; *La fontion zeta d'une monodromie*; Comment. Math. Helv. v. 50,233-248, 1975.

[2] Arnold V.I., Gusein-Zade S.M., Varchenko A.N.; *Singularities of Differentiable Maps*; Vol II. Birkhäuser, Boston-Basel-Stuttgart, 1988.

[3] Brieskorn E., Knörrer H.; *Plane curves*; Birkhäuser, Basel, Boston, Stuttgart, April 1986.

[4] Campillo A., Delgado F., Gusein-Zade S.M. ; *On the monodromy of a plane curve singularity and the Poincare series of its rings of functions*; Functional Analisis and its Applications, v.33, N 1, 56-57, 1999 .

[5] Campillo A., Delgado F., Gusein-Zade S.M. ; *The Alexander polynomial of a plane curve singularity and the ring of functions on it* ; Russian Math.Surveys, v.54, N 3, 157-158, 1999 .

[6] Casas Alvero E.; *Singularities of Plane Curves*; Cambridge University Press , 2000.

[7] Delgado F. ; *An Aritmetical factorization for the critical point set of some maps from $\mathbb{C}^2$ to $\mathbb{C}^2$.* ; In: Singularities-Lille 1991. Jean-Paul Brasselet Ed. London Mathematical Society Lecture Note Series, v.201. Cambridge Univesity Press, 61-100, 1994.

[8] Delgado F. ; *The semigroup of values of a curve singularity with several branches* ; Manuscripta Math . v.59, 347-374, 1987.

[9] de Jong T., Pfister G.; *Local Analytic Geometry*; Vieweg, Braunschweig/Wiesbaden, 2000.

[10] Eisenbud D., Neumann W.; *Three-dimensional link theory and invariants of plane curve singularities*; Ann. of Math. Studies 110, Princeton Univ. Press, Princeton, NJ, 1985.

[11] Greuel G.M., Lossen C., Shutin E.; *Singular algebraic curves*; Springer-Verlag, November 29, 2001.

[12] Greuel G.M., Pfister G., Schönemann H. ; *The computeralgebra System SINGULAR*; http://www.singular.mathematik.uni-kl.de .

[13] Lamm M.; *Hamburger-Noether-Entwicklung von Kurvensingularitäten*; Universität Kaiserslautern, August 1999.

[14] Perez Hernandez A. ;*Semigrupos de valores de curvas* ; University of Valladolid , November 18 ,1998 .

[15] Waldi R.;*Wertehalbgruppe und Singularität einer ebenen algebraischen Kurve*; Dissertation. Regensburg, 1972.

[16] Walker R.J.; *Algebraic Curves*; Springer-Verlag New York Heidelberg Berlin.

[17] Yamamoto M.; *Classification of isolated algebraic singularities by their Alexander polynomials*; Topology, v.23, N 3,277-287, 1984.

[18] Zariski O., Teissier B.; *Le problème des modules pour les branches planes*; Hermann, Paris, 1986.